ON THE CYCLICITY AND SYNTHESIS OF DIAGONAL OPERATORS ON THE SPACE OF FUNCTIONS ANALYTIC ON A DISK

Ian Deters

A Dissertation

Submitted to the Graduate College of Bowling Green
State University in partial fulfillment of
the requirements for the degree of

A DISSERTATION

May 2009

Committee:

Steven M. Seubert, Advisor

Sachi Sakthivel,
Graduate Faculty Representative

Neal L. Carothers

Gordon J. Wade

ABSTRACT

Steven M. Seubert, Advisor

A diagonal operator on the space of functions holomorphic on a disk of finite radius is a continuous linear operator having the monomials as eigenvectors. In this dissertation, necessary and sufficient conditions are given for a diagonal operator to be cyclic. Necessary and sufficient conditions are also given for a cyclic diagonal operator to admit spectral synthesis, that is, to have as closed invariant subspaces only the closed linear span of sets of eigenvectors. In particular, it is shown that a cyclic diagonal operator admits synthesis if and only if one vector, not depending on the operator, is cyclic. It is also shown that this is equivalent to existence of sequences of polynomials which seperate and have minimum growth on the eigenvalues of the operator.

This dissertation is dedicated to my Lord and Savior Jesus Christ.

Worthy are You, our Lord and our God, to receive glory and honor and power; for You created all things, and because of Your will they existed, and were created.

-Revelation 4:11

We give You thanks, O Lord God, the Alighty, Who are and Who were, because You have taken Your great power and have begun to reign.

-Revelation 11:17

Great and marvelous are Your works, O Lord God, the Almighty; Righteous and true are Your ways, King of the nations! Who will not fear, O Lord, and glorify Your name? For You alone are holy; For all the nations will come and worship before You, for Your righteous acts have been revealed.

-Revelation 15:3,4

ACKNOWLEDGMENTS

1. Soli Deo Gloria. I give thanks to Jesus Christ for giving me salvation, my friends, my family, wealth, and my mathematical ability. Surely I am blessed.

2. To Susan Jacob, Loretta Haugh, and Fred Schmitt, thank you for believing in me when I was a mediocre student. To Pat Morgan, thank you for being such a good role model, mentor, and teacher in my high school years. Thank you for boldly sharing the gospel with me.

3. Thank you David Hahn for showing me the power, beauty, and rigor of mathematics. Thank you Jeffrey Riedl for first directing my research and showing me the importance of simple examples. Thank you Sachi Sakthivel for sitting on my dissertation committee. Thank you Juan Bes, Neal Carothers, and Kit Chan for teaching me the rudiments of analysis. Thank you Alex Izzo for being a challenging teacher and an excellent expositor and putting up with my silliness. Thank you Steven Seubert for all of your advice throughout my career here. Thank you for giving me the freedom to explore topics as I saw fit. Thank you for teaching me the importance of a literature review.

4. Thank you to my parents Greg Deters, Wendy Grant, and Charles Grant (R.I.P.). Thank you for your ceaseless love and encouragement and endless supplies.

5. Thank you to Bowling Green Covenant Church. You have been a wonderful church family. You have taught me the importance of hospitality. I hope to emulate what has been modeled for me here. In particular, thank you to Michael and Beth Manahan and Michael and Nancy Sanderson. Thank you for opening up your hearts and homes to me.

6. Thank you to all of my friends who hung out with me along the way. You have made the journey quite enjoyable. In particular, thank you Steve Dinda, Sean Berger,

Ryan Vigus, Joel Manahan, Ian Manahan, Tumpa Bhattacharyya, Suzanne Miller, and Elizabeth Krumrei. Thanks to Brandon Clark, the best roomate and friend a guy can have. You have taught me a lot and you pay the bills on time.

7. Thanks to Nate Iverson for preparing the class file for this dissertation.

Table of Contents

CHAPTER 1

Spaces Of Functions Analytic In A Disk

1.1 Background And Overview Of Diagonal Operators

The theme of this document is invariant subspaces. Let X be a topological vector space and consider some continuous linear operator $T : X \to X$. A classical problem in such a situation has been to determine the closed, invariant subspaces of T. Indeed, such considerations are what give rise to theorems on canonical forms of matrices. Since the matter has already been settled in finite dimensional cases, we will concern ourselves with infinite dimensional vector spaces.

The first infinite dimensional space one usually considers is the separable Hilbert space H. The space H has a structure which closely resembles $\mathbb{C}^n$ while still being infinite dimensional. We may now consider some operator on H and attempt to determine its closed invariant subspaces. Taking inspiration from the finite dimensional case, a first approach would be to determine the eigenvectors of the operator. It is always the case that the closure of the span of some set of eigenvectors is a closed invariant subspace. However, not all operators on H have eigenvalues. For example, the forward shift operator has no eigenvalues. In fact, it is not known whether or not every operator on H has an invariant subspace. This is a famous open problem known as The Invariant Subspace Problem.

Since the Invariant Subspace Problem is generally considered to be difficult, one typically specifies a class of operators and studies the invariant subspaces of those operators. That is the approach that we will take here. In particular, we will consider an analogue of diagonal operators on H. On H a diagonal operator D is an operator such that $De_n = \lambda_n e_n$ for some $\lambda_n \in \mathbb{C}$ and some orthonormal basis $\{e_n : n \geq 1\}$. Such operators are analogues of diagonal matrices and have been well studied. In fact, compact self-adjoint operators are diagonal with respect to an appropriate basis. What is interesting is that even though diagonal operators look innocuous, they sometimes have unexpected subspaces. For instance, it is obvious that the closure of subspaces spanned by some subset of $\{e_n : n \geq 0\}$ are invariant for D. However, is that all of them? Not all of the time. In 1921, Wolff [22] gave an example of a diagonal operator with a closed invariant subspace which was not the closure of a subspace spanned by some subset of $\{e_n : n \geq 0\}$. So it seems that even diagonal operators can exhibit interesting behavior. However, there is a list of conditions given in Theorem 1.1 which are equivalent to D having only the obvious invariant subspaces.

As noted before, it is always the case that the closed linear span of some set of eigenvectors is an invariant subspace. Based on this, we make the following defintion. A continuous linear operator $T : X \to X$ is said to **admit spectral synthesis** if every closed invariant subspace M for T equals the closed linear span of the eigenvectors for T contained in M. Operators which admit spectral synthesis are called **synthetic**. Observe that an operator being synthetic is a type of minimality condition on the number of closed invariant subspaces of that operator. Loosely speaking, an operator is synthetic if the closed invariant subspaces are precisely the obvious ones. Hence, the discussion in the preceding paragraph was about which diagonal operators were synthetic and which were not.

Clearly, the subspaces X and $\{0\}$ are both invariant subspaces. Such subspaces are called trivial and are not considered of interest. What are some other invariant subspaces?

The easiest way to obtain invariant subspaces is to build them. To this end, choose some $x \in X$ such that $x \neq 0$. We shall build the smallest closed invariant subspace M containing x. What has to be in M? Certainly $x \in M$. However, since M is invariant under T, $T(M) \subseteq M$. Hence, $Tx \in M$. Also, since $Tx \in M$, $TTx = T^2x \in M$. This would imply that $TT^2x = T^3x \in M$. More generally, it must be the case that $T^n x \in M$ for $n \geq 0$. Moreover, since M is a subspace, $\text{span}(\{T^n x : n \geq 0\}) \subseteq M$. Finally, since M is closed, $\overline{\text{span}(\{T^n x : n \geq 0\})} \subseteq M$. Since M is supposed to be the smallest, closed, invariant subspace containing x and $\overline{\text{span}(\{T^n x : n \geq 0\})}$ is a closed, invariant subspace containing x, it must be that $\overline{\text{span}(\{T^n x : n \geq 0\})} = M$. Thus, it is easy to describe the smallest, closed, invariant subspaces containing any given vector x. The question becomes whether or not the subspace is trivial. Since $x \neq 0$, it is not the case that $M = \{0\}$. However, it may be the case that $M = X$.

For example, let $X = C([0,1])$, the space of continuous functions on the interval $[0,1]$ with the supremum norm. Define $T : X \to X$ by $(Tf)(x) = xf(x)$. The operator T is a continuous linear operator. However, by the Weierstrauss Approximation Theorem the smallest, closed, invariant subspace containing the function $f \equiv 1$ is X. Such behavior is common enough that it has its own name.

Given a a vector $x \in X$, the **orbit** of x under T is the set $\{T^n x : n \geq 0\}$. A vector $x \in X$ is said to be **cyclic** for T if the closed linear span of the orbit is all of X. Operators which have a cyclic vector are said to be **cyclic**. The preceding paragraph asserts that $f \equiv 1$ is a cyclic vector for T. A non-zero vector is cyclic if and only if it is not contained in a proper closed invariant subspace. Hence, the set of cyclic vectors for an operator is the complement of the union of the proper closed invariant subspaces. Finally, observe that the invariant subspace problem is asking whether or not every operator on H has a non-trivial, non-cyclic vector.

Determining when a diagonal operator on H is synthetic is a problem dating back to at least 1921. It involves the work of Wolff [22], Wermer [21], Sarason [16] and [17], Nikol'skiĭ

[12] and [13], Brown, Shields, and Zeller [2], and Sibilev [20]. The relevant parts of their work have been combined to form the following theorem.

Theorem 1.1 *Let H be a separable complex Hilbert space and let D be any bounded linear operator on H for which there exists an orthonormal basis $\{e_n\}$ for H and a sequence $\{\lambda_n\}$ of complex numbers for which $De_n = \lambda_n e_n$ for all $n \geq 0$. Then $\{\lambda_n\}$ is bounded. Moreover, D is cyclic if and only if $\lambda_m \neq \lambda_n$ for all $m \neq n$, and in this case, the following are equivalent:*

1. *The operator D admits spectral synthesis.*

2. *There does not exist a non-trivial sequence $(w_n) \in \ell^1$ for which $0 = \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$.*

3. *There does not exist a non-trivial sequence $(w_n) \in \ell^1$ for which the Wolff-Denjoy series $\sum_{n=0}^{\infty} \frac{w_n}{z-\lambda_n} = 0$ for all z with $|z| > \sup(\{|\lambda_n| : n \geq 0\})$.*

4. *There does not exist a non-trivial sequence $(w_n) \in \ell^1$ for which the complex measure $\mu \equiv \sum_{n=0}^{\infty} w_n \delta_{\{\lambda_n\}}$ consisting of point masses at the λ_n with weights w_n annihilates the polynomials.*

5. *There does not exist a non-trivial sequence $(w_n) \in \ell^1$ for which the exponential series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z} = 0$ for all $z \in \mathbb{C}$.*

6. *Every closed invariant subspace of D is invariant for the adjoint D^* of D.*

7. *The adjoint D^* of D is in the closed algebra generated by the identity operator and D in the strong operator topology.*

8. *If, in addition, the λ_n lie inside a Jordan region G and accumulate only on the boundary of G, then conditions 1 through 7 are equivalent to $\sup\{|f(z)| : z \in G\} = \sup\{|f(\lambda_n)| : n \geq 0\}$ for $f \in H(G)$ and bounded (where $H(G)$ denotes the space of functions analytic on G).*

In Chapter 4, we shall prove Theorem 4.1, an analogue of the preceding theorem for the case of diagonal operators on the space of functions analytic on a disk of finite radius. The proof of this theorem should indicate why the seemingly unrelated statements in Theorem 1.1 are equivalent. We shall then extend Theorem 4.1 result with Theorem 4.3. In particular, we shall demonstrate that spectral synthesis is equivalent to the cyclicity of one particular vector as well as the existence of sequences of polynomials which seperate, and have a minimal growth condition at, the eigenvalues of the operator.

In this document, we will consider the analogue of diagonal operators on the space of functions analytic on a disc of some finite radius and determine when cyclic operators are synthetic. In the rest of this chapter, we will record the salient features of $H(B(0, R))$. In Chapter 2, we will collect introductory information about diagonal operators and show how $H(B(0, 1))$ can be embedded into the space of diagonal operators. In Chapter 3, we will discuss the cyclicity of diagonal operators and the relationship between the cyclicity of vectors and the decay rate of their coefficients. Chapter 4 will consist in giving necessary and sufficient conditions for cyclic diagonal operators to be synthetic. We shall also demonstrate that a large family of cyclic diagonal operators is synthetic. Finally, in Chapter 5, we will construct polynomials whose existence is demonstrated in Chapter 4. Throughout we will compare and contrast our present case with that of the Hilbert space case.

1.2 The Space H_R And Convergence In Terms Of Maclu-arin Coefficients

For any $R \in (0, \infty)$, let $H_R = H(B(0, R))$ denote the space of functions analytic in the disk $B(0, R)$ of radius R centered about the origin. Let $H(\mathbb{C})$ denote the space of entire functions. We will almost always think of functions in H_R and $H(\mathbb{C})$ in terms of their power series. If $f \in H_R$ or $f \in H(\mathbb{C})$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n$, then, from the Radius of Convergence Formula, we have that $\limsup_{n\to\infty} |f_n|^{\frac{1}{n}} \leq \frac{1}{R}$ or $\limsup_{n\to\infty} |f_n|^{\frac{1}{n}} = 0$, respectively. Define a metric ρ on H_R by

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f-g\|_{R(1-\frac{1}{n})}}{1+\|f-g\|_{R(1-\frac{1}{n})}}$$

and a metric σ on $H(\mathbb{C})$ by

$$\sigma(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f-g\|_n}{1+\|f-g\|_n}$$

where $\|f-g\|_r = \sup_{|z|\leq r} |f(z)-g(z)|$ (see Conway [4] p. 142 - 152). Note that convergence in these metrics is equivalent to uniform convergence on compact sets (see Conway [4]). Moreover, both of these spaces are examples of complete, locally convex spaces. However, they are not Banach spaces since it is not always true that $\rho(cf,0) = c\rho(f,0)$. For instance, take $f \equiv 1$ and $c = 2$.

Since we will be viewing functions in H_R and $H(\mathbb{C})$ almost entirely in terms of their Maclaurin series, it will be convenient to have a test for convergence in terms of the series expansions of a sequence of functions. The criterion for convergence is the content of the next lemma and theorem.

Lemma 1.1 *Let $G \subset \mathbb{C}$ be open, $a \in G$, and $f_\alpha \in H(G)$ such that (f_α) is a net with $\lim_\alpha f_\alpha = 0$ be given. If $f_\alpha(z) = \sum_{k=0}^{\infty} f_{\alpha,k}(z-a)^k$ is f_α's expansion, then $\lim_\alpha f_{\alpha,k} = 0$ for each $k \geq 0$.*

Proof. Choose some r such that $0 < r < \operatorname{dist}(a, \partial G)$. Then (f_α) converges uniformly to 0 on $\overline{B(a,r)}$. Write $M_\alpha = \sup(\{|f_n(z)| : |z-a| \leq r\})$ and note that $M_\alpha \to 0$. Then by Cauchy's Estimate,

$$|f_{\alpha,k}| = \frac{1}{k!}|f_\alpha^{(k)}(a)| \leq \frac{1}{k!} \cdot \frac{k! M_\alpha}{r^k} \to 0.$$

□

Theorem 1.2 *Let R and $(f_n) \subset H_R$ such that $0 < R < \infty$ and $f_n(z) = \sum_{k=0}^{\infty} f_{n,k} z^k$ be given. Then $f_n \to 0$ in H_R if and only if $\lim_{n\to\infty} f_{n,k} = 0$ for each k and $\limsup_{n\to\infty} \sup(\{|f_{n,k}|^{\frac{1}{k}} : k \geq 1\}) \leq \frac{1}{R}$.*

Proof. First suppose that $\lim_{n\to\infty} f_{n,k} = 0$ for each k and $\limsup_{n\to\infty} \sup(\{|f_{n,k}|^{\frac{1}{k}} : k \geq 1\}) \leq \frac{1}{R}$. Let $E \subset B(0,R)$ be compact and $\varepsilon > 0$ be given. Write $r = \sup(\{|z| : z \in E\})$ and note that $r < R$. Choose some $t \in (\frac{1}{R}, \frac{1}{r})$, some K such that $\sum_{k>K}^{\infty} (tr)^k < \frac{\varepsilon}{2}$, and some N_1 such that $\sup(\{|f_{n,k}|^{\frac{1}{k}} : k \geq 1\}) < t$ for $n \geq N_1$. Hence, for $n \geq N_1$ and $k \geq 1$ we have that $|f_{n,k}| < t^k$. Let $r_0 = \max(\{r, 1\})$. By assumption, for fixed k we have that $|f_{n,k}| \to 0$. Choose N_2 such that $|f_{n,k}| < \frac{\varepsilon}{2(K+1)r_0^K}$ for $0 \leq k \leq K$ and $n \geq N_2$, and define $N = \max(\{N_1, N_2\})$. Then for $n \geq N$ and $z \in E$, we have that

$$|f_n(z)| \leq \sum_{k=0}^{K} |f_{n,k}| r^k + \sum_{k>K}^{\infty} |f_{n,k}| r^k \leq \sum_{k=0}^{K} |f_{n,k}| r_0^K + \sum_{k>K}^{\infty} (tr)^k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $f_n \to 0$ in H_R.

Now suppose that $f_n \to 0$ in H_R. By Lemma 1.1, we know that $\lim_{n\to\infty} f_{n,k} = 0$ for each fixed k. To show that $\limsup_{n\to\infty} \sup(\{|f_{n,k}|^{\frac{1}{k}} : k \geq 1\}) \leq \frac{1}{R}$, let $r \in (0,R)$ be given. Since $f_n \to 0$ uniformly on compact subsets of $B(0,R)$, there exists some N such that $|f_n(z)| \leq 1$ for all $z \in \overline{B(0,r)}$ whenever $n \geq N$. Then by Cauchy's Estimate, we have that $|f_{n,k}| \leq \frac{1}{r^k}$ for $k \geq 1$ whenever $n \geq N$. Hence, $|f_{n,k}|^{\frac{1}{k}} \leq \frac{1}{r}$ for $k \geq 1$ whenever $n \geq N$. Thus, $\sup(\{|f_{n,k}|^{\frac{1}{k}} : k > 0\}) \leq \frac{1}{r}$ for $n \geq N$. Since $r \in (0,R)$ was arbitrary, we have that $\limsup_{n\to\infty} \sup(\{|f_{n,k}|^{\frac{1}{k}} : k \geq 1\}) \leq \frac{1}{R}$. $\square$

The corresponding result for $H(\mathbb{C})$ was proven by Iyer [8]: In $H(\mathbb{C})$, $f_n \to 0$ if and only if $\limsup_{n\to\infty} \sup(\{|f_{n,k}|^{\frac{1}{k}} : k \geq 1\}) = 0$ and $f_{n,0} \to 0$. The above result has nice applications when combined with the following lemma.

Lemma 1.2 *Let $r \geq 0$ and $a_{n,k} \geq 0$ for $n \geq 1$ and $k \geq 0$ such that $a_{n,k} \to 0$ as $n \to \infty$ for each $k \geq 0$ be given. Then there exists a $j \geq 0$ such that $\limsup_{n\to\infty} \sup(\{a_{n,k} : k > j\}) \leq r$ if and only if for all $j \geq 0$ we have $\limsup_{n\to\infty} \sup(\{a_{n,k} : k > j\}) \leq r$.*

Proof. Necessity is trivial. For sufficiency, suppose $\limsup_{n\to\infty} \sup(\{a_{n,k} : k > j_0\}) \leq r$ for some j_0 and let $j \geq 0$ and $\varepsilon > 0$ be given. Note that if $j \geq j_0$, then $\sup(\{a_{n,k} : k > j\}) \leq \sup(\{a_{n,k} : k > j_0\})$. This implies that $\limsup_{n\to\infty} \sup(\{a_{n,k} : k > j\} \leq$

$\limsup_{n\to\infty} \sup(\{a_{n,k} : k > j_0\}) \leq r$. Now suppose that $j < j_0$. Choose N_1 such that $\sup(\{a_{n,k} : k > j_0\}) < r + \varepsilon$ for $n \geq N_1$ and N_2 such that $a_{n,k} < r + \varepsilon$ for $n \geq N_2$ and $0 \leq k \leq j_0$. Write $N = \max(\{N_1, N_2\})$ and observe that for $n \geq N$, $\sup(\{a_{n,k} : k > j\}) < r + \varepsilon$. Since ε was arbitrary we have that $\limsup_{n\to\infty} \sup(\{a_{n,k} : k > j\}) \leq r$. □

1.3 The Dual Of H_R And Of $H(\mathbb{C})$

The next result establishes the dual of H_R and $H(\mathbb{C})$ and comes from [3] p. 116.

Theorem 1.3 *1. If $0 < R < \infty$, $L \in H_R^*$, and $\ell_n = Lz^n$, then $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < R$. Conversely, if $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < R$, then there exists an $L \in H_R^*$ such that $Lf = \sum_{n=0}^{\infty} f_n \ell_n$ whenever $f(z) = \sum_{n=0}^{\infty} f_n z^n$.*

2. If $L \in H(\mathbb{C})^*$, and $\ell_n = Lz^n$, then $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < \infty$. Conversely, if $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < \infty$, then there exists an $L \in H(\mathbb{C})^*$ such that $Lf = \sum_{n=0}^{\infty} f_n \ell_n$ whenever $f(z) = \sum_{n=0}^{\infty} f_n z^n$.

Proof.

1. If $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} \geq R$, there is some subsequence (ℓ_{n_k}) such that $\lim_{k\to\infty} |\ell_{n_k}|^{\frac{1}{n_k}} \geq R$ and $|\ell_{n_k}| > 0$ for all k. Define the sequence (f_n) by $f_n = 0$ if $n \neq n_k$ for some k and $f_{n_k} = \frac{1}{\ell_{n_k}}$. Note that $\limsup_{n\to\infty} |f_n|^{\frac{1}{n}} = \limsup_{k\to\infty} |f_{n_k}|^{\frac{1}{n_k}} = \limsup_{k\to\infty} \frac{1}{|\ell_{n_k}|^{\frac{1}{n_k}}} \leq \frac{1}{R}$ and define $f \in H_R$ by $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Hence, by the continuity of L, we have that $Lf = \sum_{n=0}^{\infty} f_n \ell_n = \sum_{k=0}^{\infty} \frac{1}{\ell_{n_k}} \ell_{n_k} = \infty$. Since this would imply that $L \notin H_R^*$, we must have that $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < R$.

   Now suppose that $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < R$. Define $L : H_R \to \mathbb{C}$ by $Lf = \sum_{n=0}^{\infty} f_n \ell_n$ where $f(z) = \sum_{n=0}^{\infty} f_n z^n$. To see that L is continuous, define $L_n : H_R \to \mathbb{C}$ by $L_n f = \sum_{k=0}^{n} f_k \ell_k$ where $f(z) = \sum_{k=0}^{\infty} f_k z^k$. Clearly, $L_n \in H_R^*$ is continuous and $L_n f \to Lf$ for all $f \in H_R$. Therefore, by the Banach-Steinhaus Theorem, L is continuous.

2. The proof of part 2 of this theorem is similar to the proof of part 1. □

1.4 A Linear Homeomorphism Between H_1 and $H(G)$

Recall that the Riemann Mapping Theorem states that simply connected regions come in two categories: $\mathbb{C}$ and regions that are conformally equivalent to the disc. As a result, $B(0,1)$ and $B(0,2)$ are much more similar than $B(0,1)$ and $\mathbb{C}$. This would further suggest that H_1 and H_2 are more similar than H_1 and $H(\mathbb{C})$. The next result establishes this more generally.

Theorem 1.4 *Let G be a simply connected region such that $G \neq \mathbb{C}$ and let $f : G \to B(0,1)$ be a conformal map. Define $T : H(G) \to H_1$ by $Tg = g \circ f^{-1}$. The map T is a linear homeomorphism.*

Proof. Clearly, T is linear. Next suppose that $g_n \to g$ in $H(G)$ and $g_n \circ f^{-1} \to h$ in H_1 and let $z \in B(0,1)$ be given. Then since $g_n \to g$ in $H(G)$, $(g_n \circ f^{-1})(z) \to (g \circ f^{-1})(z)$. However, it is also the case that $(g_n \circ f^{-1})(z) \to h(z)$. Thus, $h = g \circ f^{-1} = Tg$ and T is continuous by the Closed Graph Theorem. Since f has an inverse, T is injective. Given some $h \in H_1$ observe that $h \circ f \in H(G)$ and $T(h \circ f) = h \circ f \circ f^{-1} = h$. Thus, T is surjective. Therefore, T is a homeomorphim by the Open Mapping Theorem. □

As the above theorem indicates, there are really only two types of spaces of functions analytic on a disk: H_1 and $H(\mathbb{C})$. We shall see later that these two spaces are the only important ones for questions of cyclicity and synthesis in our setting. We shall also see that the question of synthesis yields much different results for each space. The following observation is also occasionally useful.

Proposition 1.1 *Given $R_1 \leq R_2$, define $i : H_{R_2} \to H_{R_1}$ by $i(f) = f|_{B(0,R_1)}$ and $i_0 : H(\mathbb{C}) \to H_{R_1}$ by $i_0(f) = f|_{B(0,R_1)}$.*

1. *The maps i and i_0 are continuous.*

2. *If M is dense in H_{R_2} or $H(\mathbb{C})$, then $i(M)$ or $i_0(M)$ is dense in H_{R_1}.*

Proof.

1. Let $(f_n) \subseteq H_{R_2}$ and $f \in H_{R_2}$ such that $f_n \to f$ in H_{R_2} be given. Then (f_n) converges to f uniformly on compact subsets of $B(0, R_2)$. In particular, (f_n) converges to f uniformly on compact subsets of $B(0, R_1)$. Thus, $i(f_n) \to i(f)$ in H_{R_1} which implies that i is continuous. The proof for the continuity of i_0 is similar.

2. Suppose that M is dense in H_{R_2}. Then $\mathbb{C}[z] \subseteq \overline{M}$. Since i is continuous, this implies that $i(\mathbb{C}[z]) \subseteq \overline{i(M)}$. Finally, observe that since $\overline{i(\mathbb{C}[z])} = H_{R_1}$, we have that $\overline{i(M)} = H_{R_1}$. The proof for the density of $i_0(M)$ is similar. $\square$

1.5 The Strong And Weak Operator Topology On $C(H(G))$

We will have occasion later to talk about the strong and weak operator topologies on $C(H_1)$, the set of continuous linear operators which map H_1 back into H_1. Hence, we will need some precise idea about the nature of these topologies. It is actually somewhat easier to develop these notions more generally and then simply apply the results to the special case of $C(H_1)$. The results and their proofs, except for Theorem 1.5, given in this section are known. In fact some of the proofs will mimic those given in [3]. However, the references, such as [3], for them tend to be terse. As such, a fuller exposition will be given here.

Let X be a locally convex, topological vector space (in the future we shall refer to such a space as an LCS). By theorems 1.14, 1.36, and 1.37 in [15], there is some set $\mathcal{P}$ of seminorms which induce the topology on X. That is, the topology on X is the weakest topology such that p is continuous for all $p \in \mathcal{P}$. By p. 100 - 101 of [3] we may assume that $\mathcal{P}$ is closed under sums.

Denote by $C(X)$ the set of continuous, linear operators mapping X into X. For each $x \in X$, define $q_{p,x} : C(X) \to \mathbb{R}$ by $q_{p,x}(T) = p(Tx)$. Note that $q_{p,x}$ is a seminorm on $C(X)$ and define the strong operator topology (SOT) on $C(X)$ to be the toplogy with subbasis $\{q_{p,x}^{-1}(U) : p \in \mathcal{P},\, x \in X,\, U \subseteq \mathbb{R},\, U \text{ open}\}$. That is, the SOT is the weakest topology such that $q_{p,x}$ is continuous for all $x \in X$ and $p \in \mathcal{P}$. Next, for all $f \in X^*$ and $x \in X$, define $q_{f,x}$:

$C(X) \to \mathbb{R}$ by $q_{f,x}(T) = |f(Tx)|$. Once again, $q_{f,x}$ is a seminorm on $C(X)$. Define the weak operator toplogy (WOT) on $C(X)$ to be the toplogy with subbasis $\{q_{f,x}^{-1}(U) : f \in X^*,\ x \in X,\ U \subseteq \mathbb{R},\ U\text{open}\}$. That is, the WOT is the weakest topology such that $q_{f,x}$ is continuous for all $x \in X$ and $f \in X^*$. We give the basic properties of these topologies in the proposition below. In order to do so, we introduce some notation. Write $U(T_0, x_1, \ldots, x_m, p, r) = \{T : q_{p,x_k}(T - T_0) < r \text{ for } 1 \leq k \leq m\}$ and $U(T_0, x_1, \ldots, x_m, f_1, \ldots f_m, r) = \{T : q_{f_k,x_k}(T - T_0) < r \text{ for } 1 \leq k \leq m\}$.

Proposition 1.2 *Let $(T_\alpha) \subseteq C(X)$ be a net and $T \in C(X)$.*

1. *The collection*

$$\{U(T_0, x_1, \ldots, x_m, p, r) : m \geq 1, p \in \mathcal{P}, T_0 \in C(X) \text{ and } x_1, \ldots, x_m \in X\}$$

is a basis for the SOT and the collection

$$\{U(T_0, x_1, \ldots, x_m, f_1, \ldots f_m, r) : m \geq 1, T_0 \in C(X), x_1, \ldots, x_m \in X \text{ and } f_1, \ldots f_m \in X^*\}$$

is a basis for the WOT.

2. *$C(X)$ is a Hausdorff, LCS under both the SOT and the WOT.*

3. *$T_\alpha \to T$ in $C(X)$ under the SOT if and only if $T_\alpha x \to Tx$ for all $x \in X$.*

4. *$T_\alpha \to T$ in $C(X)$ under the WOT if and only if $f(T_\alpha x) \to f(Tx)$ for all $x \in X$ and $f \in X^*$.*

Proof.

1. Let an SOT neighborhood U of 0 and T_0 be given. By definition of the SOT, there are some open sets $U_1, \ldots, U_m \subseteq \mathbb{R}$, elements $x_1, \ldots, x_m \in X$, and seminorms $p_1, \ldots p_m \in \mathcal{P}$ such that $0 \in \cap_{k=1}^m q_{p_k,x_k}^{-1}(U_k) \subseteq U$. Since U_k is an open set in $\mathbb{R}$, it is a union of

open intervals. Hence, for each k, there is some interval (a_k, b_k) such that $(a_k, b_k) \subseteq U_k$ and $0 \in q^{-1}_{p_k,x_k}((a_k, b_k))$ where we allow that either $a_k = -\infty$ or $b_k = \infty$. For the sake of simplicity note that $0 \in q^{-1}_{p_k,x_k}((a_k, b_k))$ if and only if $0 \in q^{-1}_{p_k,x_k}((-\infty, b_k))$. Thus, $0 \in \cap_{k=1}^{m} q^{-1}_{p_k,x_k}((-\infty, b_k)) \subseteq U$. Write $r = \min(\{b_k : 1 \leq k \leq m\})$, $p = \sum_{k=1}^{m} p_k$, $V_0 = \cap_{k=1}^{m} q^{-1}_{p,x_k}((-\infty, r))$, and note that $0 \in V_0 \subseteq U$ and $V_0 = U(0, x_1, \ldots, x_m, p, r)$. Hence, the sets $\{U(0, x_1, \ldots, x_m, p, r) : m \geq 1, p \in \mathcal{P}$ and $x_1, \ldots, x_m \in X\}$ form a local base for 0 and their translates by T_0 form a local basis for T_0. Finally, observe that these translates are of the form prescribed in the theorem.

The result for the WOT topology follow similarly.

2. Observe that the local bases for 0 in the SOT and WOT given in part 1 are convex sets since they are induced by seminorms. To see that the SOT is Hausdorff, note that $\cap_{x \in X, p \in \mathcal{P}} q^{-1}_{p,x}(\{0\}) = \{T \in C(X) : q_{p,x}(T) = p(Tx) = 0,\, x \in X,\, p \in \mathcal{P}\} = \{0\}$. To see that the WOT is Hausdorff, note that by the Hahn-Banach theorem $\cap_{x \in X, f \in X^*} q^{-1}_{f,x}(\{0\}) = \{T \in C(X) : q_{f,x}(T) = |f(Tx)| = 0,\, x \in X,\, f \in X^*\} = \{0\}$.

3. Let a net $(T_\alpha) \subseteq C(X)$ be given and consider the SOT on $C(X)$. By [11, Prop. 2.4.4, p. 204], $T_\alpha \to T$ if and only if $p(T_\alpha x) = q_{p,x}(T_\alpha) \to q_{p,x}(T) = p(Tx)$ for all $x \in X$ and $p \in \mathcal{P}$. Hence, again by [11, Prop. 2.4.4, p. 204] and the definition of the topology on X, we see that $T_\alpha \to T$ if and only if $T_\alpha x \to Tx$.

4. The proof of this is similar to the proof of part 3 of this result. □

A few remarks are in order. First, parts 3 and 4 of the preceding proposition imply that convergence in $C(X)$ means pointwise convergence in some sense. Second, parts 3 and 4 of the above result imply that the SOT is stronger than the WOT. Moreover, if X is a Banach space, we know that the norm topology on $C(X)$ is stronger than the SOT. In general, it is not the case that these topologies coincide [3, Exercise 5, p. 276]. Third, in general it is the case that the SOT, and hence the WOT, is not metrizable [3, Prop 1.3, p. 256].

We shall now consider the closure of sets in the different topologies. To this end, we will need a few definitions. For $n \geq 1$, we shall define $X^{(n)} = X \oplus \ldots \oplus X$. That is, $X^{(n)}$ is the direct sum of X with itself $n-1$ times. Given some $T \in C(X)$, we define $T^{(n)} : X^{(n)} \to X^{(n)}$ by $T^{(n)}(x_1, \ldots, x_n) = (Tx_1, \ldots, Tx_n)$. For $T \in C(X)$ define $\text{Lat}\, T = \{M \subseteq X : T(M) \subseteq M,\ \overline{M} = M\}$. For $E \subseteq C(X)$, define $\text{Lat}\, E = \cap_{T \in E} \text{Lat}\, T$, $E^{(n)} = \{T^{(n)} : T \in E\}$, and $\text{Lat}\, E^{(n)} = \cap_{T \in E} \text{Lat}\, T^{(n)}$.

Proposition 1.3 *Functionals On $C(X)$.*

1. *Given some $f \in (X^{(n)})^*$, there exists an $f_1, \ldots f_n \in X^*$ such that $f(x_1, \ldots, x_n) = \sum_{k=1}^{n} f_k(x_k)$.*

2. *L is continuous in the SOT if and only if L is continuous in the WOT.*

Proof.

1. Define $f_k \in X^*$ by $f_k(x) = f(0, \ldots, 0, x, 0, \ldots 0)$ where x is in the kth position. It is clear that $f_k \in X^*$ and that $f(x_1, \ldots x_n) = \sum_{k=1}^{n} f_k(x_k)$.

2. Since the SOT is stronger than the WOT, it is clear that if L is continuous in the WOT, it is continuous in the SOT. To see the converse, suppose that L is continuous in the SOT. Since the SOT is a LCS whose topology is given by seminorms, by [3, Thm. 3.1, p. 108] there are seminorms $q_1, \ldots, q_k$, and positive scalers $c_1, \ldots, c_k$ such that $|L(T)| \leq \sum_{j=1}^{k} c_j q_j(T)$ for all $T \in C(X)$. By definition, for $1 \leq j \leq k$ there is some $p_j \in \mathcal{P}$ and $x \in X$ such that $q_j(T) = p_j(Tx_j)$ for all $T \in C(X)$. Define $M = \{(Tx_1, \ldots, Tx_k) : T \in C(X)\} \subseteq X^{(k)}$ and $F : M \to \mathbb{C}$ by $F(Tx_1, \ldots, Tx_k) = L(T)$. Observe that F is continuous since $|F(Tx_1, \ldots, Tx_k)| = |L(T)| \leq \sum_{j=1}^{k} c_j q_j(T) = \sum_{j=1}^{k} c_j p_j(Tx_j)$. Hence, by the Hahn-Banach Theorem, there is some $F_0 \in (X^{(k)})^*$ such that $F_0|_M = F$. By part 1, there exist $f_1, \ldots, f_k \in X^*$ such that $F_0(y_1, \ldots y_k) = \sum_{j=1}^{k} f_k(y_k)$ for all $(y_1, \ldots y_k) \in X^{(k)}$. In particular $L(T) = F(Tx_1, \ldots Tx_k) = \sum_{j=1}^{k} f_k(Tx_k)$. Therefore, L is continuous in the WOT. $\square$

Corollary 1.1 *If $E \subseteq C(X)$ is convex, then $\overline{E}^{SOT} = \overline{E}^{WOT}$.*

Proof. Since the dual of $C(X)$ under the SOT coincides with the dual of $C(X)$ under the WOT, the weak closure of E under either topology is the same. Since E is convex, by [3, Thm. 1.4, p. 126], the weak closure of E under either topology is the same as its closure. □

We shall now establish an algebraic characterization of the the closure of a subalgebra of $C(X)$.

Lemma 1.3 *If $E \subseteq C(X)$ and $T \in \overline{E}^{SOT}$, then for all $n \geq 1$, Lat $E^{(n)} \subseteq$ Lat $T^{(n)}$.*

Proof. Let $T \in \overline{E}$, $n \geq 1$, $M \in \mathrm{Lat}E^{(n)}$, and $(x_1, \ldots, x_n) \in M$ be given. Since $T \in \overline{E}$, there is some net $(T_\alpha) \subseteq E$ such that $T_\alpha \to T$. Hence, $T_\alpha x_k \to T x_k$ for $1 \leq k \leq n$. Thus, $T_\alpha^{(n)}(x_1, \ldots, x_n) = (T_\alpha x_1, \ldots, T_\alpha x_n) \to (Tx_1, \ldots, Tx_n) = T^{(n)}(x_1, \ldots, x_n)$. Also, since $M \in \mathrm{Lat}\ E^{(n)} = \cap_{S \in E} \mathrm{Lat}\ S^{(n)}$, $T_\alpha^{(n)}(x_1, \ldots, x_n) \in M$ for all α. This implies that $T^{(n)}(x_1, \ldots x_n) \in M$ since M is closed. That is $T^{(n)}(M) \subseteq M$. Hence, $M \in \mathrm{Lat}\ T^{(n)}$ and the result is proven. □

Proposition 1.4 *If $\mathcal{A} \subseteq C(X)$ is a subalgebra containing the identity I, then $\overline{\mathcal{A}}^{SOT} = \{T \in C(X) :$ Lat $\mathcal{A}^{(n)} \subseteq$ Lat $T^{(n)}$, $n \geq 1\}$.*

Proof. Note that part of this containment was proven in the preceding statement. To see the other containment, let $T \in \{S \in C(X) : \mathrm{Lat}\mathcal{A}^{(n)} \subseteq \mathrm{Lat}S^{(n)}\}$. We must show that every basic open set containing T contains an element of $\mathcal{A}$. To this end, let $x_1, \ldots, x_m \in X$, $p \in \mathcal{P}$, and $r > 0$ be given. Define $M = \overline{\{(Ax_1, \ldots, Ax_m) : A \in \mathcal{A}\}}$. Since $\mathcal{A}$ is an algebra, $M \in \mathrm{Lat}\mathcal{A}^{(m)} \subseteq \mathrm{Lat}T^{(m)}$. Also, the fact that $1 \in \mathcal{A}$ implies that $(x_1, \ldots, x_m) \in M$. Hence, $(Tx_1, \ldots, Tx_m) \in M$. Since the set $\{(Ax_1, \ldots, Ax_m) : A \in \mathcal{A}\}$ is dense in M and $(Tx_1, \ldots, Tx_m) \in M$, there is some net $(A_\alpha) \subset \mathcal{A}$ such that $(A_\alpha x_1, \ldots, A_\alpha x_m) \to (Tx_1, \ldots, Tx_m)$. Thus, $A_\alpha x_j \to T x_j$ for $1 \leq j \leq m$. Hence, there is some α such that

$p(A_\alpha x_j - Tx_j) = q_{p,x_j}(A_\alpha - T) < r$. Therefore, $A_\alpha \in U(T, x_1, \ldots, x_m, p, r)$ which implies that $T \in \overline{\mathcal{A}}$. $\square$

We now apply the above general construction to the space $X = H(G)$ where $G \subseteq \mathbb{C}$ is a region, $p_n(f) = \max\left(\left\{|f(z)| : |z| \leq n \text{ and } d(z, \mathbb{C} - G) \geq \frac{1}{n}\right\}\right)$, $d(z, \mathbb{C} - G) = \inf(\{|z - w| : w \in \mathbb{C} - G\})$, and $\mathcal{P} = \{p_n : n \geq 1\}$ (for example see [4] p. 143). As a first application of this construction, we will given a natural extension of Theorem 1.4.

Theorem 1.5 *Let G be a simply connected region such that $G \neq \mathbb{C}$ and consider $C(H(G))$ and $C(H_1)$ under the SOT. Then $C(H(G))$ and $C(H_1)$ are isomorphic.*

Proof. By Theorem 1.4, there is some $T_0 : H(G) \to H_1$ such that T_0 is a linear homeomorphism. Define $A : C(H(G)) \to C(H_1)$ by $AT = T_0 T T_0^{-1}$. Given $c \in \mathbb{C}$ and $S, T \in C(H(G))$, $A(cT + S) = T_0(cT + S)T_0^{-1} = (cT_0 T + T_0 S)T_0^{-1} = cT_0 T T_0^{-1} + T_0 S T_0^{-1} = cAT + AS$. Hence, A is linear. Let $(T_i) \subseteq C(H(G))$ be a net such that $T_i \to T$ and $f \in H(G)$ be given. By definition of the SOT and the continuity of T_0, we have that $T_i(T_0^{-1} f) \to T(T_0^{-1} f)$ and $AT_i = T_0(T_i T_0^{-1} f) \to T_0(T T_0^{-1} f) = AT$. Thus, A is continuous. Since T_0 is invertible, A is injective. Given some $S \in C(H_1)$, observe that $T_0^{-1} S T_0 \in C(H(G))$ and $A(T_0^{-1} S T_0) = S$. Hence, A is surjective. Finally, note that $A^{-1} : C(H_1) \to C(H(G))$ defined by $T_0^{-1} T T_0$ is continuous by a proof similar to the one showing the continuity of A. Therefore, A is a homeomorphism. $\square$

CHAPTER 2

Diagonal Operators

In this chapter, we define the diagonal operators. These are the main objects of study in this dissertation. In the first section, we record the basic properties of such operators. In particular, we will characterize the growth rate of the eigenvalues in terms of the domain and range of the operator. In the second section, we will observe that such operators give rise to a natural way of embedding $H(G)$ into $C(H(G))$ under the SOT when $G \neq \mathbb{C}$ is simply connected.

2.1 Definition And Basic Properties

Definition 2.1 *Let $R_1, R_2 \in (0, \infty]$ be given. A continuous linear map $D : H_{R_1} \to H_{R_2}$ whose eigenvectors contains the set $\{z^n : n \geq 0\}$ is called a* ***diagonal operator****. If $Dz^n = \lambda_n z^n$, then (λ_n) is called D's* ***associated sequence****.*

To see why these operators are called diagonal, let $D : H_{R_1} \to H_{R_2}$ be a diagonal operator with associated sequence (λ_n). Given some $f \in H_{R_1}$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$, the continuity of D implies that $(Df)(z) = D \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} f_n D z^n = \sum_{n=0}^{\infty} \lambda_n f_n z^n$. Hence, if we regard f as a column vector consisting of the coefficients of its expansion, then D can be regarded as an infinite by infinite diagonal matrix:

$$Df = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 & \dots \\ 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_0 f_0 \\ \lambda_1 f_1 \\ \lambda_2 f_2 \\ \lambda_3 f_3 \\ \vdots \end{pmatrix} = \sum_{n=0}^{\infty} \lambda_n f_n z^n.$$

Observe that Chapter 1 contains an example of a diagonal operator. To see this, let $R \in (0, \infty)$ be given and observe that the function $f : B(0, R) \to B(0, 1)$ defined by $f(z) = \frac{z}{R}$ is a homeomorphism. By Theorem 1.4, the map $T : H_R \to H_1$ defined by $Tg = g \circ f^{-1}$ is a homeomorphism. Further, observe that if $g(z) = \sum_{n=0}^{\infty} g_n z^n$, then $(Tg)(z) = (g \circ f^{-1})(z) = g(Rz) = \sum_{n=0}^{\infty} R^n g_n z^n$. That is, the operator T is a diagonal operator with associated sequence (R^n).

We now collect some basic results concerning diagonal operators. In particular, if $D : H_{R_1} \to H_{R_2}$ is a diagonal operator with associated sequence (λ_n), we shall see how R_1, R_2, and the sequence (λ_n) are related. In particular, we characterize the collection of associated sequence of diagonal operators. We start with a technical lemma that gives a certain finiteness condition on associated sequences.

Lemma 2.1 *Let $R \in (0, \infty]$ and a sequence $(\lambda_n) \subseteq \mathbb{C}$ such that $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} = \infty$ be given. Then there is some $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $\sum_{n=0}^{\infty} \lambda_n f_n z^n$ diverges for all $z > 0$.*

Proof. Since $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} = \infty$, there is some subsequence (λ_{n_k}) such that $\lim_{k\to\infty} |\lambda_{n_k}|^{\frac{1}{n_k}} = \infty$ and $|\lambda_{n_k}| > 0$ for all k. Define the sequence (f_n) by $f_{n_k} = \frac{|\lambda_{n_k}|}{\lambda_{n_k}} \cdot |\lambda_{n_k}|^{-\frac{1}{2}}$ and $f_n = 0$ if $n \neq n_k$ for all k. Note that $\limsup_{n\to\infty} |f_n|^{\frac{1}{n}} = \limsup_{k\to\infty} |f_{n_k}|^{\frac{1}{n_k}} = \limsup_{k\to\infty} \frac{1}{(|\lambda_{n_k}|^{\frac{1}{n_k}})^{\frac{1}{2}}} = 0$ and define $f \in H_R$ by $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Let $z > 0$ be given and note that

$$\sum_{n=0}^{\infty} f_n \lambda_n z^n = \sum_{k=0}^{\infty} \left(\frac{|\lambda_{n_k}|^{\frac{1}{n_k}}}{(|\lambda_{n_k}|^{\frac{1}{n_k}})^{\frac{1}{2}}} z \right)^{n_k} = \sum_{k=0}^{\infty} ((|\lambda_{n_k}|^{\frac{1}{n_k}})^{\frac{1}{2}} z)^{n_k} = \infty.$$

□

The preceding lemma shows that if the formal map $\sum_{n=0}^{\infty} f_n z^n \rightarrow \sum_{n=0}^{\infty} \lambda_n f_n z^n$ is to even be defined, much less continuous, then it must be the case that $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} < \infty$. We may now proceed to the basic properties of diagonal operators. In particular, we show that the set of eigenvalues of such an operator is necessarily $\{\lambda_n : n \geq 0\}$ where (λ_n) is its associated sequence.

Proposition 2.1 *Let $D : H_{R_1} \rightarrow H_{R_2}$ be a diagonal operator with associated sequence (λ_n). Then*

1. *the set of eigenvalues of D is $\{\lambda_n : n \geq 0\}$,*

2. *the set of eigenvectors for λ_n is the set $\overline{span(\{z^k : \lambda_k = \lambda_n\})}$,*

3. *if $R_1 < R_2 = \infty$, then $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} = 0$, and*

4. *if $R_1 < \infty$ and $R_2 < \infty$, then $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq \frac{R_1}{R_2}$.*

Proof.

1. Suppose that for some $f \in H_{R_1}$ there is some $\lambda \in \mathbb{C}$ such that $Df = \lambda f$ and $f \neq 0$. Write $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and observe that $0 = \lambda f(z) - (Df)(z) = \sum_{n=0}^{\infty} (\lambda - \lambda_n) f_n z^n$. Then $(\lambda - \lambda_n) f_n = 0$ for all $n \geq 0$ and there is some n such that $f_n \neq 0$ since $f \neq 0$. Thus, $\lambda = \lambda_n$ for some n. Conversely, if $f(z) = z^n$, then $(Df)(z) = \lambda_n z^n = \lambda_n f(z)$. Hence, $\{\lambda_n : n \geq 0\}$ is the set of eigenvalues of D.

2. Suppose that for some $f \in H_{R_1}$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^k$, $Df = \lambda_n f$. Then $0 = \lambda_n f(z) - (Df)(z) = \sum_{k=0}^{\infty} (\lambda_n - \lambda_k) f_k z^k$. Since $(\lambda_n - \lambda_k) f_k = 0$ for all $k \geq 0$, $\lambda_k = \lambda_n$ for all k such that $f_k \neq 0$. That is, $f(z) = \sum_{k, \lambda_k = \lambda_n} f_k z^k \in \overline{\text{span}(\{z^k : \lambda_k = \lambda_n\})}$. Conversely, if $f \in H_{R_1} \cap \overline{\text{span}(\{z^k : \lambda_k = \lambda_n\})}$ then $f(z) = \sum_{k, \lambda_k = \lambda_n} f_k z^k$. Hence, $\lambda_n f(z) = \sum_{k, \lambda_k = \lambda_n} \lambda_n f_k z^k = \sum_{k, \lambda_k = \lambda_n} \lambda_k f_k z^k = (Df)(z)$.

3. Suppose $R_1 < \infty = R_2$ and define the function $f \in H_{R_1}$ by $f(z) = \sum_{n=0}^{\infty} \frac{1}{R_1^n} z^n$. Since $Df \in H(\mathbb{C})$, we have that $0 = \limsup_{n\to\infty} \left|\frac{1}{R_1^n}\lambda_n\right|^{\frac{1}{n}} = \frac{1}{R_1} \limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}}$. Hence, $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} = 0$.

4. Suppose $R_1, R_2 < \infty$ and define the function $f \in H_{R_1}$ by $f(z) = \sum_{n=0}^{\infty} \frac{1}{R_1^n} z^n$. Since $Df \in H_{R_2}$, we have that $\frac{1}{R_2} \geq \limsup_{n\to\infty} \left|\frac{1}{R_1^n}\lambda_n\right|^{\frac{1}{n}} = \frac{1}{R_1} \limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}}$. Hence, $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq \frac{R_1}{R_2}$. $\square$

The preceding proposition demonstrates that the associated sequence of a diagonal operator has a growth condition which is specified in terms of its domain and codomain. The following result is a sort of converse to this. Namely, if a sequence $(\lambda_n) \in \mathbb{C}$ is such that $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} < \infty$, then there exists a diagonal operator with associated sequence (λ_n).

Proposition 2.2 *Let $R_1 \in (0, \infty]$ and a sequence $(\lambda_n) \subseteq \mathbb{C}$ such that $R \equiv \limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} < \infty$ be given. Choose $R_2 \in \left(0, \frac{R_1}{R}\right]$ (where we interpret $\frac{R_1}{R} = \infty$ when $R = 0$). Then the map $D : H_{R_1} \to H_{R_2}$ defined by $Df = g$, where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} \lambda_n f_n z^n$, is a diagonal operator.*

Proof. We first show that such an operator is well-defined. First suppose that $R_1 < \infty$ and let $f \in H_{R_1}$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be given. Then

$$\limsup_{n\to\infty} |\lambda_n f_n|^{\frac{1}{n}} \leq (\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}})(\limsup_{n\to\infty} |f_n|^{\frac{1}{n}}) \leq \frac{R}{R_1} \leq \frac{1}{R_2}$$

where we take $\frac{1}{R_2} = 0$ if $R_2 = \infty$. Hence, $Df = g \in H_{R_2}$ where $g(z) = \sum_{n=0}^{\infty} \lambda_n f_n z^n$. The proof when $R_1 = \infty$ follows similarly.

Since it is clear that D is linear and has the monomials $\{z^n : n \geq 0\}$ as eigenvectors, we need only prove the continuity of D. To this end, for each $n \geq 1$ let $f_n, f \in H_{R_1}$ and $g \in H_{R_2}$ such that $f_n \to f$ and $Df_n \to g$ as $n \to \infty$ be given. Write $f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k, f(z) = \sum_{k}^{\infty} a_k z^k$, and $g(z) = \sum_{k=0}^{\infty} g_k z^k$. By Lemma 1.1, we have that $\lim_{n\to\infty} a_{n,k} = a_k$ and

$\lambda_k a_k = \lim_{n\to\infty} \lambda_k a_{n,k} = g_k$. Thus, $g(z) = \sum_{k=0}^{\infty} g_k z^k = \sum_{k=0}^{\infty} \lambda_k a_k z^k = (Df)(z)$. Thus, $Df = g$ and D is continuous by the Closed Graph Theorem. $\square$

A consequence of the preceding propositions is that the eigenvalues of a diagonal operator may be unbounded. For instance, if $\lambda_n = 4^n$ for $n \geq 0$, then $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} = 4$. Hence, the diagonal operator $D : H(\mathbb{C}) \to H(\mathbb{C})$ with associated sequence (λ_n) has unbounded eigenvalues. This is in contrast to the spectral theory of operators on Banach spaces since such operators have compact spectrums.

We also note that above results indicate that it is possible to have different operators $D : H_{R_1} \to H_{R_2}$ and $D' : H_{R_3} \to H_{R_4}$ with the same associated sequence (λ_n). The action of such operators is formally the same but defined on different spaces. We will make use of this observation in subsequent chapters. Since we will be interested in the cyclicity of vectors, we now give a simple result concerning orbits of vectors and intertwining operators. For ease of notation, for $f \in H_R$ and D a diagonal operator, write $\text{span}(\text{orb}_D(f)) = \text{span}(\{D^n f : n \geq 0\})$.

Proposition 2.3 *Given R_1, R_2 such that $0 < R_1, R_2 \leq \infty$, let $D : H_{R_1} \to H_{R_1}$, $D' : H_{R_1} \to H_{R_2}$, and $\overline{D} : H_{R_2} \to H_{R_2}$ be diagonal operators with associated sequences (λ_n), (λ_n'), and (λ_n), respectively.*

1. *If $p \in \mathbb{C}[z]$, then $D'p(D) = p(\overline{D})D'$. In particular, $D'D = \overline{D}D'$.*

2. *If $f \in H_{R_1}$, then $D'(\text{span}(\text{orb}_D(f))) = \text{span}(\text{orb}_{\overline{D}}(D'f))$.*

Proof.

1. Let $f \in H_{R_1}$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be given. Then

$$(D'Df)(z) = \sum_{n=0}^{\infty} \lambda_n' \lambda_n f_n z^n = \sum_{n=0}^{\infty} \lambda_n \lambda_n' f_n z^n = (\overline{D}D'f)(z).$$

Hence, $D'D = \overline{D}D'$. Let $p(z) \in \mathbb{C}[z]$ where $p(z) = \sum_{k=0}^{n} c_k z^k$ be given. Then

$$\begin{aligned} D'(p(D)) &= D'\left(\sum_{k=0}^{n} c_k D^k\right) = \sum_{k=0}^{n} c_k D' D^k = \sum_{k=0}^{n} c_k D^{*k} D' \\ &= \left(\sum_{k=0}^{n} c_k D^{*k}\right) D' = (p(\overline{D}))D'. \end{aligned}$$

2. Note that if $x \in \operatorname{span}(\operatorname{orb}_D(f))$, then there is some $n \geq 0$ and $a_0, \dots, a_n$ such that $x = \sum_{k=0}^{n} a_k D^k f = p(D)f$ where $p(z) = \sum_{k=0}^{n} a_k z^k$. Hence, by the first part of this proposition, we have that

$$\begin{aligned} D'(\operatorname{span}(\operatorname{orb}_D(f))) &= D'(\{p(D)f : p(z) \in \mathbb{C}[z]\}) = \{D'p(D)f : p(z) \in \mathbb{C}[z]\} \\ &= \{p(\overline{D})D'f : p(z) \in \mathbb{C}[z]\} = \operatorname{span}(\operatorname{orb}_{\overline{D}}(D'f)). \quad \square \end{aligned}$$

2.2 A Natural Embedding Of H_1 Into $C(H_1)$

In this section, we use diagonal operators to embed H_1 into $C(H_1)$ in the SOT. To motivate this, observe that if $D : H_1 \to H_1$ is a diagonal operator with associated sequence (λ_n), then $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq 1$ by Proposition 2.1. Hence, $f \in H_1$ where $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$. Conversely, given some $f \in H_1$ where $f(z) = \sum_{n=0}^{\infty} f_n z^n$, the Radius of Convergence Formula implies that $\limsup_{n\to\infty} |f_n|^{\frac{1}{n}} \leq 1$. Thus, there is some diagonal operator $D : H_1 \to H_1$ with associated sequence (f_n). In light of this, it seems that there is little distinction between functions and diagonal operators acting on those functions. We make the following definition.

Definition 2.2 *Given R such that $0 < R \leq \infty$ and $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$, define $R' = \limsup_{n\to\infty} |f_n|^{\frac{1}{n}}$ and choose $R_1 \in (0, \infty]$ and $R_2 \in \left(0, \frac{R_1}{R'}\right]$ (where $\frac{R_1}{R'} = \infty$ if $R' = 0$). Define $D_f : H_{R_1} \to H_{R_2}$ to be the diagonal operator which has associated sequence (f_n).*

For the rest of this document, let $u \in H_1$ be defined by $u(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. A simple computation shows that if $g \in H_1$, then $g = D_g u$. This computation is a crucial observation and makes the idea that there is little distinction between diagonal operators and analytic functions more precise. As a first application of this idea, we will show that H_1 can be naturally embedded in $C(H_1)$.

Proposition 2.4 *Let $\mathcal{D} \subseteq C(H_1)$ denote the set of diagonal operators on H_1. For each $k \geq 0$ define $\pi_k : H_1 \to H_1$ by $(\pi_k f)(z) = f_k z^k$ where $f(z) = \sum_{n=0}^{\infty} f_n z^n$.*

1. *The subspace $\mathcal{D}$ is SOT closed.*

2. *The map $f \to D_f$ is a linear homeomorphism.*

3. *The set $\{\pi_n : n \geq 0\}$ has a dense linear span in $\mathcal{D}$.*

Proof.

1. This follows from condition 3 of Proposition 1.2 and Lemma 1.1 by choosing $x = f$ where $f(z) = z^k$.

2. It is clear that the map $f \to D_f$ is linear. To see that it is continuous, for $n \geq 1$, let $f_n, f \in H_1$ such that $f_n \to f$ be given. Let $h \in H_1$ such that $h(z) = \sum_{n=0}^{\infty} h_n z^n$ be given. Then

$$D_{f_n} h = D_{f_n} D_h u = D_h D_{f_n} u = D_h f_n \to D_h f = D_h D_f u = D_f D_h u = D_f h.$$

   Thus, $D_{f_n} \to D_f$ in the SOT.

   To see that the inverse is continuous, let $(D_\alpha) \subseteq C(H_1)$ and $D \in C(H_1)$ such that (D_α) is a net of diagonal operators having associated sequences $(\lambda_{\alpha,k})$ and $D_\alpha \to D$ be given. By part 1 of this result, D is a diagonal operator with associated sequence $(\lambda_k = \lim_\alpha \lambda_{\alpha,k})$. Define $f_\alpha(z) = \sum_{k=0}^{\infty} \lambda_{\alpha,k} z^k$ and $f(z) = \sum_{k=0}^{\infty} \lambda_k z^k$. Then $D_\alpha g \to Dg$

for all $g \in H_1$. In particular, the preceding statement is true for u. Thus, $f_\alpha = D_\alpha u \to Du = f$.

3. This follows from part 2 of this result, the fact that $\overline{\text{span}(\{z^k : k \geq 0\})} = H_1$, and the observation that $z^k \to \pi_k$. $\square$

Note that the above result combined with Theorem 1.5 implies that there is a natural way to embed $H(G)$ into $C(H(G))$ in the SOT where G is a simply connected region and $G \neq \mathbb{C}$. Moreover, the embedding of $H(G)$ in $C(H(G))$ is closed. Finally, observe that the above shows that diagonal operators have a canonical quality to them. In a sense, any function analytic on a simply-connected region which is not $\mathbb{C}$ *is* a diagonal operator.

CHAPTER 3

Cyclic Diagonal Operators

In this chapter, we consider the cyclic diagonal operators. In the first section, we give necessary and sufficient conditions for a diagonal operators to be cyclic. In the second section, we examine a connection between cyclic vectors for one operator and cyclic vectors for a related operator on a different space. Loosely speaking, we also show that the faster a vector's coefficients decay, the better chance it has of being cyclic. In the third section, we discuss the existence of common cyclic vectors.

3.1 Definition Of And Criteria For Cyclicity

Recall from Section 1.1 that a vector x in a complete metrizable topological vector space X is said to be **cyclic** for a continuous linear operator $T : X \to X$ on X if the closed linear span of the orbit $\{T^n x : n \geq 0\}$ of x under T is all of X. If x is not cyclic, then $\overline{\text{span}(\text{orb}_T(x))}$ is an example of a non-trivial, closed, invariant subspace for T. Since we are concerned with diagonal operators and their closed, invariant subspaces, it would be useful to have a characterization of which operators are cyclic. The characterization of which diagonal operators are cyclic is precisely the same as it is in the Hilbert space case (see [9, Cor. 6]). This is the content of the next theorem. Before we state the theorem, recall that a set is residual if it is the complement of a set of first category.

Theorem 3.1 *Let $D : H_R \to H_R$ and $R \in (0, \infty)$ be a diagonal operator with associated sequence (λ_n).*

1. *If $f \in H_R$ is cyclic for D and $f(z) = \sum_{n=0}^{\infty} f_n z^n$, then $f_n \neq 0$ for all $n \geq 0$.*

2. *The operator D is cyclic if and only if $\lambda_n \neq \lambda_m$ when $n \neq m$.*

3. *If D is cyclic, then the set of cyclic vectors is residual in H_R.*

Proof.

1. Suppose that $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and there is some n_0 such that $f_{n_0} = 0$. Define $L : H_R \to \mathbb{C}$ by $Lg = \sum_{n=0}^{\infty} \ell_n g_n$ where $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $\ell_n = 0$ for $n \neq n_0$, and $\ell_{n_0} = 1$. By Theorem 1.3, $L \in H_R^*$. Also, $L(\text{span}(\text{orb}_D(f))) = \{0\}$ for $k \geq 0$ by construction. Since L is a non-trivial functional, $\text{span}(\text{orb}_D(f))$ can not be dense by the Hahn-Banach theorem.

2. Suppose there is some pair $m, n \geq 0$ such that $\lambda_n = \lambda_m$ and $m \neq n$. Choose $f \in H_R$ such that $f(z) = \sum_{k=0}^{\infty} f_k z^k$ and $f_k \neq 0$ for $k \geq 0$. Define $L : H_R \to \mathbb{C}$ by $Lg = \sum_{k=0}^{\infty} \ell_k g_k$ where $g(z) = \sum_{k=0}^{\infty} g_k z^k$, $\ell_k = 0$ for $k \notin \{m, n\}$, $\ell_m = 1$, and $\ell_n = \frac{-f_m}{f_n}$. By Theorem 1.3, $L \in H_R^*$. Also, L is non-trivial and $L(\text{span}(\text{orb}_D(f))) = \{0\}$ for $j \geq 0$ by construction. Thus, $\text{span}(\text{orb}_D(f))$ can not be dense. By part 1 of this theorem, this implies that D is not cyclic.

   Now suppose that D has distinct eigenvalues and define $D' : H(\mathbb{C}) \to H(\mathbb{C})$ to be the diagonal operator with associated sequence (λ_n). By Proposition 2.1, D' is continuous. Marin and Seubert [10] proved that D' is cyclic. Hence, there is some $f \in H(\mathbb{C})$ such that for each $k \geq 0$, there is some sequence of polynomials $(p_n) \subseteq \mathbb{C}[z]$ such that $p_n(D')f \to z^k$ in $H(\mathbb{C})$. Define $i : H(\mathbb{C}) \to H_R$ by $i(f) = f|_{B(0,R)}$. Thus, by propostion 1.1, $p_n(D)i(f) \to z^k$ in H_R. Since $z^k \in \overline{\text{span}(\text{orb}_D(i(f)))}$ for $k \geq 0$, it is the case that $H_R = \overline{\text{span}(\text{orb}_D(i(f)))}$. Therefore, D is cyclic and $i(f)$ is a cyclic vector.

3. Suppose that D is cyclic. Define $D' : H(\mathbb{C}) \to H(\mathbb{C})$ to be the diagonal operator with associated sequence (λ_n) and $i : H(\mathbb{C}) \to H_R$ by $i(f) = f|_{B(0,R)}$. By Proposition 2.2, D' is continuous. Marin and Seubert [10] proved that D' has a dense set of cylic vectors C. From the proof of the previous part of this theorem, every element of $i(C)$ is cyclic for D. Moreover, since C is dense in $H(\mathbb{C})$, by Proposition 1.1, $i(C)$ is dense in H_R. Write $\mathcal{O} = \{p(D) : p \in \mathbb{C}[z]\}$. Since $\overline{\{Tf : T \in \mathcal{O}\}} = H_R$ for each $f \in i(C)$ and $i(C)$ is dense in H_R, it follows from Theorem 1 in [6] that the set of cyclic vectors is residual. $\square$

The above result was proven by Deters and Seubert [5] in 2007 and by Bernal-González, Calderón-Moreno*, and Prado-Bassas [1] in 2006. However, the above proof is different from both of these. Bernal-González et. al. also observe in their paper that no diagonal operator is supercyclic or hypercyclic. They also note that any vector space of cyclic vectors, excluding the zero vector, necessarily has dimension one.

3.2 Cyclicity Of Vectors In Terms Of Coefficient Decay Rates

As noted after Propostion 2.2, it is possible to have two diagonal operators $D : H_{R_1} \to H_{R_2}$ and $D' : H_{R_3} \to H_{R_4}$ with the same associated sequence. The following two results examine how the cyclic vectors of each operator are related to the other. To aid in this, we now define the operator which appeared as an example at the beginning of Chapter 2. It is the natural homeomorphism between H_R and H_1 when $R \in (0, \infty)$.

Definition 3.1 *Given $0 < R < \infty$, define $D_R : H_R \to H_1$ where D_R has associated sequence (R^n).*

Proposition 3.1 *Given R_1, R_2 such that $0 < R_1, R_2 \leq \infty$, let $D : H_{R_1} \to H_{R_1}$, $D' : H_{R_1} \to H_{R_2}$, and $\overline{D} : H_{R_2} \to H_{R_2}$ be diagonal operators with associated sequences (λ_n), (λ'_n), and*

(λ_n) respectively. Suppose also that $\lambda'_n \neq 0$ for all $n \geq 0$. If f is a cyclic vector for D, then $D'f$ is a cyclic vector for $\overline{D}$.

Proof. Let $k \geq 0$ be given. Since f is cyclic for D, there is some sequence $(p_n) \in \mathbb{C}[z]$ such that $p_n(D)f \to z^k$ as $n \to \infty$. By Proposition 2.3, we have that $D'(\text{span}(\text{orb}_D(f))) = \text{span}(\text{orb}_{\overline{D}}(D'f))$. Hence, $D'(p_n(D)f) \in \text{span}(\text{orb}_{\overline{D}}(D'f))$ and $D'(p_n(D)f) \to D'z^k = \lambda'_k z^k \in \overline{\text{span}(\text{orb}_{\overline{D}}(D'f))}$ as $n \to \infty$. Since $\lambda'_k \neq 0$, we have that $z^k \in \overline{\text{span}(\text{orb}_{\overline{D}}(D'f))}$. Since k was arbitrary and the monomials have a dense linear span in H_{R_2}, $\text{span}(\text{orb}_{\overline{D}}(D'f))$ is dense in H_{R_2}. That is $D'f$ is cyclic for $\overline{D}$. □

Theorem 3.2 *Given R such that $0 < R < \infty$, let $D : H_R \to H_R$ and $D' : H_1 \to H_1$ be diagonal operators with associated sequence (λ_n). Then f is cyclic for D if and only if $D_R f$ is cyclic for D'.*

Proof. Let $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be given. Let $\overline{D} : H_1 \to H_R$ be the diagonal operator with associated sequence $\left(\frac{1}{R^n}\right)$. If f is cyclic for D, then $D_R f$ is cyclic for D' by the Proposition 3.1. Similarly, if $D_R f$ is cyclic for D', then $f = \overline{D} D_R f$ is cyclic for D by the Proposition 3.1. □

We have already observed that if $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq 1$, then a diagonal operator with associated sequence (λ_n) can be defined on H_R where $0 < R < \infty$. The above result combined with the fact that D_R is a homeomorphism essentially says that with respect to which vectors are cyclic, we need only direct our attention at H_1. Hence, the only interesting spaces in which to study cyclicity are H_1 and $H(\mathbb{C})$.

Given a cyclic operator D, we know that D has a residual set of cyclic vectors. However, it is not clear which vectors are cyclic. The following set of results show that we need not investigate each vector seperately for cyclicity. Rather, once we are able to establish the cyclicity of a particular vector, we are then able to conclude that a whole class of related vectors is cyclic. These results are cast in terms of the decay rates of the coefficients of the vectors.

Theorem 3.3 *Given R such that $0 < R \leq \infty$, let $D : H_R \to H_R$ be a diagonal operator with associated sequence (λ_n). Suppose that $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is cyclic for D and let $g \in H_R$ such that $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_n \neq 0$ for all $n \geq 0$ be given. If $\limsup_{n\to\infty} \left(\frac{|g_n|}{|f_n|}\right)^{\frac{1}{n}} \leq 1$, then g is cyclic for D.*

Proof. Note that since f is cyclic for D, we have that $f_n \neq 0$ for all $n \geq 0$. Since $\limsup_{n\to\infty} \left(\frac{|g_n|}{|f_n|}\right)^{\frac{1}{n}} \leq 1$ we may define $D' : H_R \to H_R$ to be the diagonal operator which has $(\frac{g_n}{f_n})$ as its associated sequence. Then $D'f = g$ and g is cyclic by Proposition 3.1. □

The content of the following corollary is that the cyclicity of vectors is related to the decay rate of the coefficients of the given vector. In particular, once a particular decay rate yields a cyclic vector, then all decay rates less than it also yield a cyclic vector.

Corollary 3.1 *Given R such that $0 < R \leq \infty$, let $D : H_R \to H_R$ be a diagonal operator with associated sequence (λ_n). Let $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$.*

1. *If f is cyclic for D and $g \in H_R$ such that $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $0 < |g_n| \leq |f_n|$ for all $n \geq 0$, then g is cyclic for D.*

2. *f is cyclic for D if and only if g is cyclic for D where $g(z) = \sum_{n=0}^{\infty} |f_n| z^n$.*

Proposition 3.2 *Suppose that D is a diagonal operator on H_1 with associated sequence (λ_n), $f, g \in H_1$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n, g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_n \neq 0$ for $n \geq 0$, $0 < \liminf_{n\to\infty} |f_n|^{\frac{1}{n}} = R$, and $\limsup_{n\to\infty} |g_n|^{\frac{1}{n}} = R'$. Let $R_0 \in \left(0, \min\left(\{1, \frac{R}{R'}\}\right)\right]$ (where we interpret $\frac{R}{R'} = \infty$ in case $R' = 0$) and let $D' : H_{R_0} \to H_{R_0}$ be the diagonal operator with associated sequence (λ_n). If f is cyclic for D, then $g|_{B(0,R_0)}$ is cyclic for D'.*

Proof. Observe that $\limsup_{n\to\infty} \left(\frac{|g_n|}{|f_n|}\right)^{\frac{1}{n}} \leq \frac{\limsup_{n\to\infty} |g_n|^{\frac{1}{n}}}{\liminf_{n\to\infty} |f_n|^{\frac{1}{n}}} = \frac{R'}{R}$. Define $\overline{D} : H_1 \to H_{R_0}$ to be the diagonal operator with associated sequence $\left(\frac{g_n}{f_n}\right)$ and note that $D_0 f = g|_{B(0,R_0)}$. By Proposition 3.1, since f is cyclic for D, $g|_{B(0,R_0)}$ is cyclic for D'. □

The next theorem is important because it establishes two important facts about cyclic vectors. The first is that sometimes it suffices to check the cyclicity of one vector to determine the cyclicity of all possible cyclic vectors. The second fact is that every cyclic operator has a decay rate associated with it. If the coefficients of some vector decay slower than this decay rate, then they are not cyclic.

Theorem 3.4 *Suppose that D is a diagonal operator on H_1 and $g \in H_1$ such that $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $g_n \neq 0$ for $n \geq 0$, and $R \equiv \limsup_{n\to\infty} |g_n|^{\frac{1}{n}}$.*

1. *Suppose that $f \in H_1$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$, $f_n \neq 0$ for $n \geq 0$, and we have that $\liminf |f_n|^{\frac{1}{n}} = 1$. If f is cyclic, then g is cyclic.*

2. *Suppose that $f_n \in H_1$ such that $f_n(z) = \sum_{k=0}^{\infty} f_{n,k} z^n$, $f_{n,k} \neq 0$ for $n, k \geq 0$, $0 < \liminf |f_{n,k}|^{\frac{1}{k}} = R_n$, and $\lim_{n\to\infty} R_n = R$. If f_k is cyclic for all k, then g is cyclic.*

Proof.

1. This follows immediately from the preceding proposition.

2. Define $R'_n = \min\left(\left\{1, \frac{R_n}{R}\right\}\right)$ and a sequence of diagonal operators (D_n) such that $D_n : H_{R'_n} \to H_{R'_n}$ and each D_n has the same associated sequence as D. Define $i_n : H_1 \to H_{R'_n}$ by $i_n(h) = h|_{B(0,R'_n)}$. Then from Proposition 3.2, $i_n(g)$ is a cyclic vector for D_n for all $n \geq 0$. That is, $\text{span}(\text{orb}_{D_n}(i_n(g)))$ is dense in $H_{R'_n}$ for all $n \geq 0$. Note that since D_n and D have the same associated sequence for $n \geq 0$, $\text{span}(\text{orb}_{D_n}(i_n(g))) = i_n(\text{span}(\text{orb}_D(g)))$. Write $M = \text{span}(\text{orb}_D(g))$. To see that M is dense in H_1, let $L \in H_1^*$ be given such that $L(M) = \{0\}$. Then by Theorem 1.3, there is some sequence (ℓ_k) such that $\limsup_{k\to\infty} |\ell_k|^{\frac{1}{k}} < 1$ and if $f \in H_1$ where $f(z) = \sum_{n=0}^{\infty} f_k z^k$, then $Lf = \sum_{k=0}^{\infty} f_k \ell_k$. Since $\limsup_{k\to\infty} |\ell|^{\frac{1}{k}} < 1$, there is some n such that $\limsup_{k\to\infty} |\ell|^{\frac{1}{k}} < R'_n$. Define $L' \in H_{R'_n}^*$ by $L'f = \sum_{k=0}^{\infty} f_k \ell_k$ for $f \in H_{R'_n}$ where $f(z) = \sum_{k=0}^{\infty} f_k z^k$. Since, by Proposition 1.1, $i_n(M)$ is dense in $H_{R'_n}$ and $L'(i_n(M)) = L(M) = \{0\}$, we have that $L' = 0$ by the Hahn-Banach Theorem. Thus, $\ell_k = 0$ for all k. Hence, $L = 0$ and M is dense in H_1. That is, g is cyclic for D. $\square$

Proposition 3.3 *Write* $E = \{f \in H_1 : f(z) = \sum_{n=0}^{\infty} f_n z^n$ *and* $\limsup_{n\to\infty} |f_n|^{\frac{1}{n}} = 1\}$. *The set* E *is residual in* H_1.

Proof. For $n \geq 1$, define $i_n : H_{1+\frac{1}{n}} \to H_1$ by $i_n(h) = h|_{B(0,1)}$ and $E_n = h_n\left(H_{1+\frac{1}{n}}\right)$. Observe that $E \cap E_n = \emptyset$ for $n \geq 1$, $H_1 = E \cup (\cup_{n=1}^{\infty} E_n)$, and E_n is of first category in H_1 by the Open Mapping Theorem. Therefore, E is residual in H_1. $\square$

To make the preceding result more concrete, for each $R \in (0, 1]$, define $f_R \in H_1$ by $f_R(z) = \sum_{n=0}^{\infty} R^n z^n = \frac{1}{1-Rz}$ and let $D : H_1 \to H_1$ be a cyclic diagonal operator. By Theorem 3.1, a necessary condition for $g \in H_1$ to be cyclic is that it must have non-zero coefficients. There are two possibilities. Either f_1 is cyclic or it is not. If it is, then by part 1 of Theorem 3.4, g is cyclic. That is, the converse of part 1 of Theorem 3.1 holds for D.

Now suppose that f_1 is not cyclic, but $f_{R'}$ is cyclic for some $R' \in (0, 1)$. Then by part 2 of Theorem 3.4 if $R \leq R'$ or $R > R'$, then f_R is cyclic or f_R is not cyclic, respectively.

3.3 Common Cyclic Vectors For Families Of Diagonal Operators

We say that a vector x in a complete metrizable topological vector space X is a **common cyclic vector** for a set $\mathcal{T}$ of cyclic operators on X if x is a cyclic vector for each operator T in $\mathcal{T}$. Herrero has shown that a cyclic operator on a Banach space has a dense set of cyclic vectors if and only if the point spectrum of its adjoint has empty interior (see [7, Thm. 1, p. 918]). Moreover, Shields has shown that the set of cyclic vectors of an operator on a Banach space is a $\mathcal{G}_\delta$ set (see [19, Prop. 40, p. 411]). Hence by the Baire Category Theorem any countable collection of cyclic operators on a Banach space the point spectra of all of whose adjoints have empty interior has a dense set of common cyclic vectors.

In this section, we show that the uncountable collection of cyclic diagonal operators on H_R each of whose eigenvalues are separated (in a sense made precise below) has a dense set of common cyclic vectors. Hence, even though H_R is not a Banach space, a conclusion, similar to the one in the Banach space setting, may be reached.

Theorem 3.5 *Let* $\mathcal{D}_0$ *denote the collection of cyclic diagonal operators on* H_R *each of whose set of eigenvalues* $\{\lambda_n : n \geq 0\}$ *is such that* $\inf \{|\lambda_i - \lambda_j| : i \neq j\} > 0$. *Then* $\mathcal{D}_0$ *has a residual set of common cyclic vectors.*

Proof. For each $D \in \mathcal{D}_0$, define $\overline{D} : H(\mathbb{C}) \to H(\mathbb{C})$ to be the diagonal operator with the same associated sequence as D. Define $\overline{\mathcal{D}} = \{\overline{D} : D \in \mathcal{D}_0\}$. By Theorem 10 in [10], $\overline{\mathcal{D}}$ has a dense set $\mathcal{C}$ of common cylic vectors. Define $i : H(\mathbb{C}) \to H_R$ by $i(f) = f|_{B(0,R)}$. By Proposition 1.1, i is continuous and $i(\mathcal{C})$ is dense in H_R. By the proof of part 2 of Theorem 3.1, for all $f \in i(\mathcal{C})$ and $D \in \mathcal{D}_0$, f is a cyclic vector for D. Write $\mathcal{O} = \{p(D) : p \in \mathbb{C}[z], D \in \mathcal{D}_0\}$. Since $\overline{\{Tf : T \in \mathcal{O}\}} = H_R$ for each $f \in i(\mathcal{C})$ and $i(\mathcal{C})$ is dense in H_R, it follows from Theorem 1 in [6] that the set of common cyclic vectors is residual. $\square$

It is not known if the set of all cyclic diagonal operators on H_R has a common cyclic vector.

CHAPTER 4

Synthetic Diagonal Operators

In this chapter we investigate the closed invariant subspaces of cyclic diagonal operators. This leads naturally to determining which operators are synthetic. The definition of a synthetic operator as well as equivalent conditions for synthesis are given in section one. Of particular interest will be the fact that synthetic diagonal operators satisfy the converse of part 1 of Theorem 3.1. Part 1 of Theorem 3.1 states that in order for a vector to be cyclic, it must have all non-zero coefficients. It is shown in Theorem 4.1 that if a diagonal operator is synthetic, then a vector having all non-zero coefficients is also a sufficient condition for cyclicity. It is also shown that an operator being synthetic makes connections between the linear independence of an exponential series related to the operator, closure of the algebra generated by the operator in the SOT, and the existence of sequences of polynomials which have a very delicate growth condition at the eigenvalues of the operator. In the second section, we will give examples of synthetic diagonal operators. Throughout this chapter, recall that the function $u \in H_1$ is defined by $u(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. Also, for each $k \geq 0$, $\pi_k \in C(H_1)$ is defined by $(\pi_k f)(z) = f_k z^k$ where $f(z) = \sum_{n=0}^{\infty} f_n z^n$.

4.1 Definition Of And Equivalences For Synthesis

Now that we know which operators are cyclic, we may now proceed to finding the closed, invariant of such operators. To this end, recall that a continuous linear operator $T : X \to X$ on a complete metrizable topological vector space X is said to **admit spectral synthesis** if every closed invariant subspace M for T equals the closed linear span of the eigenvectors for T contained in M. Operators which admit spectral synthesis are called synthetic. By definition, a diagonal operator on H_R having associated sequence (λ_n) has as eigenvectors the monomials z^n. If D is cyclic, then the eigenvalues are distinct and scaler multiples of the monomials are the only eigenvectors for D (see Theorem 3.1). Hence a cyclic diagonal operator on H_R admits spectral synthesis if and only if the lattice of closed invariant subspaces of D consists precisely of the closed linear span of sets $\{z^n : n \in N\}$ of monomials where N is an arbitrary subset of nonnegative integers.

Theorems 4.1 and 4.3 of this section gives various equivalent conditions for a cyclic diagonal operator on H_R to admit spectral synthesis where $0 < R < \infty$. While Theorem 4.1 is very much an analogue of Proposition 5 in [10], Theorem 4.3 demonstrates that there are big differences between diagonal operators on $H(\mathbb{C})$ and diagonal operators on H_1.

We begin with two technical lemmas.

Lemma 4.1 *Let M be any closed subspace of H_R other than the whole space H_R or $\{0\}$ and define K to be the set of nonnegative integers k for which there exists an $f \in M$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ in with $f_k \neq 0$. Then there exists a $g \in M$ such that $g(z) = \sum_{n=0}^{\infty} g_n z^n$ with $g_k \neq 0$ for all k in K.*

Proof. By means of contradiction, suppose that no such function in M exists. Then $M = \cup_{k\in K} M_k$ where $M_k \equiv \{h(z) \equiv \sum_{r=0}^{\infty} a_r z^r \in M : a_k = 0\}$. Observe that $M_k = M \cap \pi_k^{-1}(\{0\})$, which implies that M_k is closed in M. Define $i_k : M_k \to M$ by $i_k(f) = f$. Since M is closed in H_R, it is complete, and hence of second category in M. If $i_k(M_k)$ was of second category in M, then by the Open Mapping Theorem $M_k = i(M_k) = M$. This is not the case by the definition of K. Thus, M_k is of first category in M for all $k \in K$. Hence, $\cup_{k \in K} M_k = M$ is of first category in M which contradicts the Baire Category Theorem. $\square$

Lemma 4.2 *Suppose that* $\left(\frac{|a_n|}{n}\right)$ *is not bounded, then for all* r*, there is some sequence* (b_n) *which depends on* R *such that* $\limsup |b_n|^{\frac{1}{n}} = 0$ *and* $\sum_{n=0}^{\infty} b_n e^{a_n z}$ *does not converge in* H_R*.*

Proof. If $\left(\frac{|a_n|}{n}\right)$ is not bounded, then there is some subsequence $\left(\frac{|a_{n_k}|}{n_k}\right)$ such that $\frac{|a_{n_k}|}{n_k} \geq k$ for all k. By setting $b_n = 0$ for n not in the above subsequence, we may suppose that $n^2 \leq |a_n|$ for all n. This implies that $|\operatorname{Re} a_n| \geq \frac{1}{2}n^2$ or $|\operatorname{Im} a_n| \geq \frac{1}{2}n^2$ for all n. Hence, there is some subsequence (a_{n_k}) such that either $|\operatorname{Re} a_{n_k}| \geq \frac{1}{2}k^2$ for all k or $|\operatorname{Im} a_{n_k}| \geq \frac{1}{2}k^2$ for all k. Without loss of generality, suppose that $|\operatorname{Re} a_{n_k}| \geq \frac{1}{2}k^2$ for all k. By setting $b_n = 0$ for n not in the sequence, we may suppose that $|\operatorname{Re} a_n| \geq \frac{1}{2}n^2$ for all n. Thus, there is some subsequence (a_{n_k}) such that either $\operatorname{Re} a_{n_k} \geq \frac{1}{2}k^2$ for all k or $\operatorname{Re} a_{n_k} \leq -\frac{1}{2}k^2$ for all k. By setting $b_n = 0$ for n not in the subsequence, without loss of generality, we may suppose that $\operatorname{Re} a_n \geq \frac{1}{2}n^2$ for all n or $\operatorname{Re} a_n \leq -\frac{1}{2}n^2$ for all n. Choose x such that $0 < x < R$, write $s = \frac{|\operatorname{Re} a_n|}{\operatorname{Re} a_n}$, $c_n = \frac{|e^{a_n s x}|}{e^{a_n s x}}$, and $b_n = \frac{1}{n^n} c_n$. Then $\limsup |b_n|^{\frac{1}{n}} = 0$ and

$$\sum_{n=0}^{\infty} b_n e^{a_n s x} = \sum_{n=0}^{\infty} \frac{1}{n^n} |e^{a_n s x}| = \sum_{n=0}^{\infty} \frac{1}{n^n} e^{x s \operatorname{Re} a_n} \geq \sum_{n=0}^{\infty} \frac{1}{n^n} e^{x \frac{1}{2} n^2} = \sum_{n=0}^{\infty} \left(\frac{1}{n} e^{x \frac{1}{2} n}\right)^n = \infty.$$

If it is the case that $|\operatorname{Im} a_{n_k}| \geq \frac{1}{2}k^2$ for all k, then the preceding proof will work with a minor adjustment in the choice of x. $\square$

The following theorem establishes the first set of equivlences for the synthesis of cyclic diagonal operators on H_R. Of these conditions, we highlight two of them for further investigation. Condition 3 will allow us to develop some more useful equivalences and condition 5 will allow us to demonstrate large classes of synthetic, cyclic diagonal operators.

Theorem 4.1 *Let* $R \in (0, \infty)$ *and* D *be the cyclic diagonal operator on* H_R *having associated sequence* (λ_n)*. Then the following are equivalent:*

1. *The operator* D *admits spectral synthesis.*

2. *Every closed invariant subspace of D is the closed linear span of $\{z^n : n \in N\}$ where N is an arbitrary set of nonnegative integers.*

3. *Every function $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $f_n \neq 0$ for all $n \geq 0$ is cyclic for D.*

4. *There does not exist a non-trivial sequence $(w_n) \subseteq \mathbb{C}$ for which $\limsup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$.*

5. *If, in addition, $\left\{\frac{\lambda_n}{n} : n \geq 1\right\}$ is bounded, then $g \in H_\varepsilon$ where $g(z) = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ whenever $(w_n) \subseteq \mathbb{C}$ for which $\limsup |w_n|^{1/n} < 1$ and $\epsilon = [\ln(1/\limsup |w_n|^{1/n})]/[\sup(\{|\lambda_n|/n : n \geq 1\})]$. There does not exist a non-trivial sequence $(w_n) \subseteq \mathbb{C}$ for which $\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < 1$ and $0 = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ for all $z \in B(0, \epsilon)$.*

Proof. .

1. $1 \Leftrightarrow 2$: This equivalence was demonstrated in the remarks at the beginning of the chapter.

2. $2 \Rightarrow 3$: Let $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $f_n \neq 0$ for all $n \geq 0$ be given. Since $f_n \neq 0$ for all $n \geq 0$, we have by part 2 that the only closed invariant subspace for D containing f also contains that set $\{z^n : n \geq 0\}$. Hence, the only closed, invariant subspace for D containing f is H_R. That is, f is cyclic for D.

3. $3 \Rightarrow 2$: Let M be an arbitrary closed invariant subspace for D other than the empty set or $\{0\}$. Define K to be the set of nonnegative integers k for which there exists a function $f \in M$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ in M with $f_k \neq 0$. Clearly M is a subset of the closed linear span of $\{z^k : k \in K\} \equiv M_0$. By way of contradiction, assume that $M \neq M_0$. Then there is some $g \in M_0$ and some $L \in H_R^*$ such that $L(M) = \{0\}$ and $L(g) \neq 0$. Define $\ell_n = L(z^n)$ and $L^* H_R \to \mathbb{C}$ by $L^* h = \sum_{n=0}^{\infty} \ell_n^* h_n$

where $h(z) = \sum_{n=0}^{\infty} h_n z^n$, $\ell_k^* = \ell_k$ for each $k \in K$, and $\ell_n^* = 0$ for each $n \notin K$. Note that $\limsup_{n\to\infty} |\ell_n^*|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} < R$ so that $L^* \in H_R^*$. Also, observe that, by construction, $L^*(g) \neq 0$. By Lemma 4.1, there is some $s \in M$ such that $s(z) = \sum_{n=0}^{\infty} s_n z^n$ where $s_n \neq 0$ when $n \in K$. Define s^* to be the function $s^*(z) = \sum_{n=0}^{\infty} s_n^* z^n$ where $s_n^* = s_n$ when $n \in K$ and $s_n^* = \frac{1}{R^n}$ when $n \notin K$. Thus, $s^* \in H_R$ and by hypothesis, s^* is cyclic for D which implies that $\operatorname{span}(\operatorname{orb}_D(s^*))$ is dense in H_R. However, by construction, $L^*(D^k s^*) = L(D^k s) = 0$ for all $k \geq 0$. Since L^* is non-trivial, this would imply that $\operatorname{span}(\operatorname{orb}_D(s^*))$ is not dense in H_R. Since this is a contradiction, $M = M_0$.

4. $3 \Rightarrow 4$: Let $f \in H_R$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $f_n \neq 0$ for $n \geq 0$ and $L \in H_R^*$ be given. Write $w_n = f_n L(z_n)$ and observe that $\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |f_n|^{\frac{1}{n}} \cdot \limsup_{n\to\infty} |L(z_n)|^{\frac{1}{n}} < \frac{1}{R} R = 1$. Moreover, for $k \geq 0$, we have that $L(D^k f) = \sum_{n=0}^{\infty} \lambda_n^k f_n L(z^n) = \sum_{n=0}^{\infty} w_n \lambda_n^k$. The vector f is cyclic for D by hypothesis. Thus, if $L(D^k f) = 0$ for $k \geq 0$, then $L = 0$ by the Hahn-Banach Theorem.

5. $4 \Rightarrow 3$. Suppose that there is some non-trivial sequence (w_n) such that $\sum_{n=0}^{\infty} w_n \lambda_n^k = 0$ for $k \geq 0$ and $\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < 1$. Define $\ell_n = w_n R^n$, $f \in H_R$ by $f(z) = \sum_{n=0}^{\infty} \frac{1}{R^n} z^n$, and $L \in H_R^*$ by $Lg = \sum_{n=0}^{\infty} \ell_n g_n$ where $g(z) = \sum_{n=0}^{\infty} g_n z^n$. Then by construction L is non-trivial and $L(D^k f) = \sum_{n=0}^{\infty} \ell_n \lambda_n^k \frac{1}{R^n} = \sum_{n=0}^{\infty} w_n \lambda_n^k = 0$ for $k \geq 0$. Thus, f is not cyclic.

6. $4 \Leftrightarrow 5$: Let $\left\{\frac{\lambda_n}{n} : n \geq 1\right\}$ be bounded and let $(w_n) \subseteq \mathbb{C}$ be any sequence for which $\limsup |w_n|^{1/n} < 1$. Then the series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ converges uniformly and absolutely on every compact subset of the open ball $B(0, \epsilon)$ by the Root Test where $\epsilon = [-\ln \limsup_{n\to\infty} |w_n|^{\frac{1}{n}}]/[\sup(\{\frac{|\lambda_n|}{n} : n \geq 1\})]$. Hence, $g \in H_\varepsilon$ where $g(z) = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$. Moreover, $g^{(k)}(0) = \sum_{n=0}^{\infty} w_n \lambda_n^k$ and so $\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0$ for all z in the open ball $B(0, \varepsilon)$ if and only if $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$. The result follows. $\square$

The extra hypothesis preceding condition 5 of Theorem 4.1 is needed in light of the lemma preceding the theorem.

Concerning condition 4 of Theorem 4.1, in 1921, Wolff [22] gave the first example of a non-trivial sequence $(w_n) \subseteq \mathbb{C}$ and a sequence $(\lambda_n) \subseteq \mathbb{C}$ of distinct complex numbers for which $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$. In Wolff's example, the sequence (λ_n) is bounded (and so $\limsup |\lambda_n|^{1/n} \leq 1$) and (w_n) is in ℓ^1. In 1952, Wermer showed that the condition $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ is equivalent to the operator D on a separable complex Hilbert space $\mathcal{H}$ diagonalizable with respect to an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ and satisfying $De_n = \lambda_n e_n$ for all $n \geq 0$ failing to admit spectral synthesis (see [21, Thm. 1, p. 270]). The full set of equivalent conditions was given in Theorem 1.1

Note that it is the precise rate of decay of the coefficients (w_n) occuring in the condition $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ that is different from Theorem 1.1 to Theorem 4.1. For instance, condition 4 of Theorem 4.1, which pertains to diagonal operators on H_R, requires that $\limsup |w_n|^{1/n} < 1$ whereas the analogous condition 2 of Theorem 1.1, which pertains to diagonal operators on Hilbert spaces, only requires the weaker condition that $(w_n) \in \ell_1$.

Observe that while we have defined what the dual of H_R is as a vector space, we have not placed any topology on it. As such, it is difficult to speak of the continuity of the adjoint of an operator D. What the precise nature of H_R^* is will have to first be answered before a condition equivalent to condition 7 in Theorem 1.1 can be given.

Regarding condition 3 in Theorem 1.1, the study of Wolff-Denjoy series has a long and rich history. Of particular interest has been conditions for an analytic function to be representable as a Wolff-Denjoy series, and conditions for such a representation, if one exists, to be unique. Borel, Beurling, and Carleman all gave sufficient conditions for the representation of an analytic function as a Wolff-Denjoy series to be unique in terms of the rate of decay of the coefficients in the representing series. Sibilev in 1995 gave a definitive uniqueness theorem of this type (see Sibilev [20]). Wolff-Denjoy series have also been studied extensively by Poincare, Wolff, Borel, Carleman, and Beurling, amongst others, mainly in connection with quasianalyticity and analytic continuation (see the recent monograph of Ross and Shapiro

[14]).

Wolff's example of a nontrivial sequence $(w_n) \in \ell^1$ and bounded sequence of distinct complex numbers (λ_n) for which $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ has been extended to sequences $\{\lambda_n\}$ of distinct complex numbers which are unbounded. For instance, in 1936, Natanson showed that there exists a sequence $(w_n) \subseteq \mathbb{C}$ for which $\sum_{n=0}^{\infty} |w_n||\lambda_n|^k < \infty$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ in the special case $\lambda_n = n$ for all $n \geq 0$ (see 5.7.8c(v) on p. 128 of Nikol' skiĭ [12]). In 1959, Makarov generalized Natanson's example to include any sequence $(\lambda_n)\mathbb{C}$ of complex numbers for which $|\lambda_n| \to \infty$ (see 5.7.8c(vi) on p. 128 of Nikol' skiĭ [12]).

However, we will see as consequences of Corollary 4.4 and Theorem 4.5 below that the coefficients (w_n) which occur in Wolff's example and in Natanson's example fail to satisfy the condition that $\limsup |w_n|^{1/n} < 1$. In fact, it remains an open question as to whether or not every cyclic diagonal operator on H_R admits spectral synthesis. That is, it is not known if there exists a sequence (λ_n) of distinct complex numbers for which $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq 1$ and a nontrivial sequence $(w_n) \subseteq \mathbb{C}$ for which $\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < 1$ with $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$.

We now give a simple, but useful, reduction concerning the problem of spectral synthesis.

Theorem 4.2 *Let $D : H_R \to H_R$, $D' : H_1 \to H_1$, and $\overline{D} : H(\mathbb{C}) \to H(\mathbb{C})$ be diagonal operators with associated sequence (λ_n).*

1. *The operator D is synthetic if and only if D' is synthetic.*

2. *If D' is synthetic, then $\overline{D}$ is synthetic.*

Proof. Let $f \in H_R$, $g \in H_1$, and $h \in H(\mathbb{C})$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n, g(z) = \sum_{n=0}^{\infty} g_n z^n$, and $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $f_n, g_n, h_n \notin \{0\}$ for all $n \geq 0$ be given.

1. Define $D_0 = D_R^{-1}$. Observe that Df and $D_0 g$ both have all non-zero coefficients. By Theorem 4.1 condition 3, if D is synthetic, then $D_0 g$ is cyclic for D. Hence, by Propostion 3.1 $(D_R D_0)g = g$ is cyclic for D'. Since g was arbitrary, by condition 3 of Theorem 4.1, D' is synthetic. Similarly, if D' is synthetic, then D is synthetic.

2. If D' is synthetic, then by condition 3 of Theorem 4.1, u is a cyclic vector for D'. Hence, by Proposition 3.1, $(D_h u) = h$ is cyclic for $\overline{D}$. Since h was arbitrary, by condition 3 of Theorem 4.1, $\overline{D}$ is synthetic. $\square$

The above results show that all questions of synthesis can be reduced to studying whether or not some operator is synthetic on H_1 or on $H(\mathbb{C})$. Moreover, we know that if some operator is synthetic on H_1, then it is synthetic on $H(\mathbb{C})$. As a first application of this result we give some more equivalences for a diagonal operator to be synthetic. Before we do, recall that $\mathcal{D} \subseteq C(H_1)$ denotes the subspace of diagonal operators on H_1.

Theorem 4.3 *Let $D : H_1 \to H_1$ be a cyclic diagonal operator. Then the following are equivalent:*

1. *The operator D is synthetic.*
2. *The function u is cyclic.*
3. *There is some $f \in H_1$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ where $f_n \neq 0$ for $n \geq 0$, $\liminf |f_n|^{\frac{1}{n}} = 1$, and f is cyclic.*
4. *There is some sequence $(f_n) \in H_1$ such that $f_n(z) = \sum_{k=0}^{\infty} f_{n,k} z^k$ where $f_{n,k} \neq 0$ for $n, k \geq 0$, $0 < \liminf |f_{n,k}|^{\frac{1}{k}} = R_n$, $\lim_{n\to\infty} R_n = 1$, and f_n is cyclic for all n.*
5. *For each $j \geq 0$, there is some sequence of polynomials $(p_n) \subset \mathbb{C}[z]$, depending on j, such that $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\limsup_{n\to\infty} \sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) \leq 1$.*
6. *Let $\mathcal{A}$ be the algebra generated by D and the identity. That is $\mathcal{A} = span(\{D^n : n \geq 0\})$. In the SOT, $\mathcal{D} = \overline{\mathcal{A}}$.*
7. *Let $\mathcal{A}$ be the algebra generated by D and the identity. In the SOT, $\pi_k \in \overline{\mathcal{A}}$ for all $k \geq 0$.*

Proof.

1. $1 \Rightarrow 2$: This follows from part 3 of Theorem 4.1 and the fact that $u(z) = \sum_{n=0}^{\infty} z^n$.

2. $2 \Rightarrow 3$: This follows by hypothesis and the fact that $u(z) = \sum_{n=0}^{\infty} z^n$.

3. $3 \Rightarrow 4$: This follows by hypothesis, by defining $f_n = f$, where f is the function whose existence is given.

4. $4 \Rightarrow 1$: This follows from part 2 of Theorem 3.4 and part 3 of Theorem 4.1.

5. $2 \Leftrightarrow 5$: Note that u is cyclic for D if and only if for each $j \geq 0$, we have that $z^j \in \overline{\text{span}(\text{orb}_D(u))}$. Let $j \geq 0$ be given. The function $z^j \in \overline{\text{span}(\text{orb}_D(u))}$ if and only if there is some sequence $(p_n) \subseteq \mathbb{C}[z]$ such that $p_n(D)u \to z^j$. Observe that $(p_n(D)u)(z) = \sum_{k=0}^{\infty} p_n(\lambda_k) z^k$. By Theorem 1.2 and Lemma 1.2 $p_n(D)u \to z^j$ if and only if $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\limsup_{n\to\infty} \sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) \leq 1$.

6. $7 \Rightarrow 2$: First suppose that $\pi_k \in \overline{\mathcal{A}}$ for all $k \geq 0$. Let $j \geq 0$ be given. Since $\pi_j \in \overline{\mathcal{A}}$ and $\mathcal{D}$ is metrizable by Proposition 2.4, there is some sequence $(x_n) \in \mathcal{A}$ such that $\lim_{n\to\infty} x_n = \pi_j$. Since $x_n \in \mathcal{A}$, for each n there some $p_n \in \mathbb{C}[z]$ such that $x_n = p_n(D)$. Since $C(H_1)$ has been endowed with the SOT, $p_n(D)u = x_n u \to \pi_j u = z^j$. Hence, $z^j \in \overline{\text{span}(\text{orb}_D(u))}$. Since j was arbitrary, we have that $z^j \in \overline{\text{span}(\text{orb}_D(u))}$ for all $j \geq 0$. Hence, $\overline{\text{span}(\text{orb}_D(u))} = H_1$. That is, u is cyclic.

7. $2 \Rightarrow 7$: Let $j \geq 0$ be given. Since u is cyclic, there is a sequence of polynomials (p_n) such that $p_n(D)u \to z^j = \pi_j u$. Let $g \in H_1$ be given and recall that $D_g u = g$. Thus,

$$p_n(D)g = p_n(D)D_g u = D_g p_n(D)u \to D_g z^j = g_j z^j = \pi_j g.$$

Since $g \in H_1$ was arbitrary, we have that $p_n(D)g \to \pi_j g$ for all $g \in H_1$. Also, the fact that $p_n(D) \in \mathcal{A}$ for all n yields that $\pi_j \in \overline{\mathcal{A}}$. Since $j \geq 0$ was arbitrary, we have that $\pi_j \in \overline{\mathcal{A}}$ for all $j \geq 0$.

8. $6 \Leftrightarrow 7$: This follows immediately from part 6 of this theorem and part 3 of Proposition 2.4. $\square$

Some remarks are in order. The first regards condition 2. That D is synthetic is a statement about the closed invariant subspaces of D. In general, knowing all of the closed invariant subspaces of an operator is equivalent to knowing all of the cyclic vectors of the operator. This follows from the fact that f is cyclic if and only if f is not in any proper closed invariant subspace. However, cyclic diagonal operators give examples where one needs to know substantially less. In particular, one needs only to check whether or not the function u is cyclic. If it is, then one knows all of the closed, invariant subspaces. If it is not cyclic, then one knows that there exists some closed, invariant subspace which is not the closure of the span of some set of monomials. In particular, one knows that the subspace $\overline{\text{span}(\text{orb}_D(u))}$ is non-trivial.

Second, note that a consequence of condtion 4 is that if D is not synthetic, then there is some $0 \leq R < 1$ such that if $f \in H_1$, $f(z) = \sum_{n=0}^{\infty} f_n z^n$, and $R \leq \liminf |a_n|^{\frac{1}{n}}$, then f is not cyclic. Note that this R represents a type of upper bound on how fast the coefficients of f decay. Hence, another way to say this is that, if D is not synthetic and the coefficients of f decay too slowly, then f is not a cyclic vector for D.

Third, condition 5 is useful because it gives a purely computational approach to determining synthesis which can potentially take advantage of the literature written on polynomials. In fact, the above result could be strengthened slightly. To see this, let $p \in \mathbb{C}[z]$ be given and write $n = deg(p)$. Then there is some $M > 0$ such that $|p(z)| \leq \max(\{M, M|z|^n\})$ for all $z \in \mathbb{C}$. If $(\lambda_k) \subseteq \mathbb{C}$ is a sequence such that $\limsup_{k\to\infty} |\lambda_k|^{\frac{1}{k}} \leq 1$, then we have that $\limsup_{k\to\infty} |p(\lambda_k)|^{\frac{1}{k}} \leq \limsup_{k\to\infty} \max(\{M^{\frac{1}{k}}, M^{\frac{1}{k}}(|\lambda_k|^{\frac{1}{k}})^n\}) = 1$. Hence, $\sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) = \max_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\})$.

Observe that the growth condition on the polynomials is potentially delicate. If it is the case that $\{\lambda_n : n \geq 0\}$ is unbounded and $p \in \mathbb{C}[z]$ is nonconstant, then the set $\{p(\lambda_n) : n \geq 0\}$ is unbounded. Hence, $\sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) > 1$. Therefore, $\limsup_{n\to\infty} \sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) \geq$

1. In other words, the polynomials satisfy a minimal type of growth condition.

Fourth, while condition 7 is clearly weaker than condition 6, condition 6 has been included because it is interesting. As was already noted in Propostion 2.4, $\mathcal{D}$ is closed in the SOT. Thus, $\overline{\mathcal{A}} \subseteq \mathcal{D}$ for any diagonal operator D. It is only in the case of a synthetic diagonal operator D that $\overline{\mathcal{A}} = \mathcal{D}$.

It should be noted that what really contributes to the proof of Theorem 4.3 is the observation that we should concentrate our attention on the function u. It is the universal nature of this function that yields the various results. This is in contrast to diagonal operators on $H(\mathbb{C})$. In this case, it is not known if any such universal vectors exist.

Before producing some examples of synthetic operators, it will be useful to talk about diagonal operators formed from a subsequence of a diagonal operator. That is, given a diagonal operator D with associated sequence (λ_k) and a subsequence (λ_{k_j}) such that $\limsup_{j\to\infty} |\lambda_{k_j}|^{\frac{1}{j}} \leq 1$, there is a diagonal operator D' with associated sequence (λ_{k_j}). A natural problem is to determine if D' is synthetic when it is known that D is synthetic. We shall start with a simple lemma.

Lemma 4.3 *Let $D : H_R \to H_R$ be a diagonl operator and $a, b \in \mathbb{C}$ be given such that $a \neq 0$ and $R \in \{1, \infty\}$. Then D is synthetic if and only if $aD + b$ is synthetic.*

Proof. Note that if M is a closed invariant subspace of D, then M is a closed invariant subspace of $aD + b$ and conversely. Thus, all of the closed, invariant subspaces of D are spanned by the monomials if and only if all of the closed, invariant subspaces of $aD + b$ are spanned by the monomials. $\square$

Proposition 4.1 *Let $D : H_R \to H_R$ be a diagonal operator with associated sequence (λ_n) where $R \in \{1, \infty\}$. Suppose (λ_{n_k}) is such that there is some $M > 0$ where $n_k \leq Mk$ for $k \geq 1$ and define $D' : H_R \to H_R$ to be the diagonal operator with associated sequence (λ_{n_k}). If D is synthetic, then D' is synthetic.*

Proof. We shall prove this by contraposition for the case $R = 1$. The case $R = \infty$ follows similarly. First, note that the operator D' is defined since $\limsup_{k\to\infty} |\lambda_{n_k}|^{\frac{1}{k}} = \limsup_{k\to\infty} (|\lambda_{n_k}|^{\frac{1}{n_k}})^{\frac{n_k}{k}} \leq \limsup_{k\to\infty} \max(\{(|\lambda_{n_k}|^{\frac{1}{n_k}})^M, 1\}) = 1$. If D' is not synthetic, then there is some sequence $(w_k) \subseteq \mathbb{C}$ such that $\limsup_{k\to\infty} |w_k|^{\frac{1}{k}} < 1$ and $0 = \sum_{k=0}^{\infty} w_k \lambda_{n_k}^j$ for $j \geq 0$. Define $\gamma_n = w_k$ if $n = n_k$ for some k and $\gamma_n = 0$ otherwise. Then $\sum_{n=0}^{\infty} \gamma_n \lambda_n^j = \sum_{k=0}^{\infty} w_k \lambda_{n_k}^j = 0$ for all $j \geq 0$. Also, $\limsup_{n\to\infty} |\gamma_n|^{\frac{1}{n}} = \limsup_{k\to\infty} |w_k|^{\frac{1}{n_k}}$
$= \limsup_{k\to\infty} (|w_k|^{\frac{1}{k}})^{\frac{k}{n_k}} \leq \limsup_{k\to\infty} (|w_k|^{\frac{1}{k}})^{\frac{1}{M}} < 1.$ □

We now present a result which says that including or discarding a finite number of eigenvalues does not change whether or not the operator is synthetic. To this end, suppose that σ is a permutation of the nonnegative integers. It is immediate from condition 4 of Theorem 4.3 that $D : H_R \to H_R$ with associated sequence (λ_k) is synthetic if and only if $D' : H_R \to H_R$ with associated sequence $(\lambda_{\sigma(k)})$ is synthetic as long as both operators are defined. Hence, without loss of generality, we may assume that the eigenvalues we are including or discarding are the first eigenvalues in the associated sequence.

Proposition 4.2 *Let $D : H_R \to H_R$ be a diagonal operator with assoicated sequence (λ_n) and choose $\lambda \notin \{\lambda_n : n \geq 0\}$.*

1. *Let $D' : H_R \to H_R$ be a diagonal operator with associated sequence (λ'_n) such that $\lambda'_0 = \lambda$ and $\lambda'_n = \lambda_{n-1}$ for $n \geq 1$. If D is synthetic, then D' is synthetic.*

2. *Let $D' : H_R \to H_R$ be a diagonal operator with associated sequence (λ'_n) such that $\lambda'_n = \lambda_{n+1}$ for $n \geq 0$. If D is synthetic, then D' is synthetic.*

Proof.

1. Suppose that D' is not synthetic. Then there is some w and (w_n) such that $\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < 1$ and $0 = w\lambda^k + \sum_{n=0}^{\infty} w_n \lambda_n^k$ for $k \geq 0$. Thus, for $k \geq 1$ we have that

$$\sum_{n=0}^{\infty} w_n(\lambda_n-\lambda)^k = \sum_{n=0}^{\infty} w_n \sum_{j=0}^{k}\binom{k}{j}\lambda_n^j(-\lambda)^{k-j} = \sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\sum_{n=0}^{\infty} w_n\lambda_n^j$$
$$= \sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}(-w\lambda^j) = -w\lambda^k\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}1^j$$
$$= -w\lambda^k((-1)+1)^k = 0.$$

Define $\gamma_n = w_n(\lambda_n-\lambda)$. Since $\limsup_{n\to\infty}|\lambda_n|^{\frac{1}{n}} \leq 1$, we have that $\limsup_{n\to\infty}|\lambda_n - \lambda|^{\frac{1}{n}} \leq 1$. Thus, $\limsup_{n\to\infty}|\gamma_n|^{\frac{1}{n}} < 1$. and $\sum_{n=0}^{\infty}\gamma_n(\lambda_n-\lambda)^k = 0$ for all $k \geq 0$. Hence, $D-\lambda$ is not synthetic. This implies that D is not synthetic.

2. This is a corollary to the previous proposition. □

4.2 Examples Of Synthetic Operators

In this section, we shall present two classes of examples of synthetic operators. These classes, combined with Lemma 4.3, will produce a large class of synthetic operators.

Theorem 4.4 *Every cyclic diagonal operator D on H_R whose eigenvalues $\{\lambda_n : n \geq 0\}$ are bounded admits spectral synthesis.*

Proof. By means of contradiction, assume that D is a cyclic diagonal operator on H_R whose eigenvalues $\{\lambda_n : n \geq 0\}$ are bounded but which fails spectral synthesis. Without loss of generality, by Lemma 4.3, we my assume that $|\lambda_n| < 1$ for all $n \geq 0$. By condition 4 of Theorem 4.1, there exists a non-trivial sequence $(w_n) \subseteq \mathbb{C}$ for which $\limsup|w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n\lambda_n^k$ for all $k \geq 0$. Observe that the series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ converges uniformly on compact subsets of $\mathbb{C}$. To see this, note that if $R > 0$ is given and $M = \sup(\{|\lambda_n| : n \geq 0\})$, then $|\sum_{n=0}^{\infty} w_n e^{\lambda_n z}| \leq \sum_{n=0}^{\infty}|w_n|e^{MR} < \infty$ for $|z| \leq R$. Define $g \in H(\mathbb{C})$ by $g(z) = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$. Since $0 = \sum_{n=0}^{\infty} w_n\lambda_n^k = g^{(k)}(0)$ for all $k \geq 0$, we have that

$g(z) = 0$ for all complex numbers z. Also, observe that since $|\lambda_n|^{\frac{1}{n}} < 1$, there is some $\gamma \in (\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}}, 1)$ and $c > 0$ such that $|w_n| \le c\gamma^n$ for all $n \ge 0$. In particular, $(w_n) \in \ell^1$. Hence by Proposition 2 of Sibilev [20, p. 147], $0 \equiv \sum_{n=0}^{\infty} w_n/(z-\lambda_n)$ whenever $|z| > 1$. Since $\sum_{n=0}^{\infty} \frac{\ln(c\gamma^n)}{n^2} = \sum_{n=0}^{\infty} \frac{\ln c}{n^2} + \sum_{n=0}^{\infty} \frac{\ln \gamma}{n} = -\infty$, we have that $w_n \equiv 0$ for all $n \ge 0$ by the theorem on p. 146 of Sibilev [20], a contradiction. $\square$

It follows from Theorem 4.4 that there exist cyclic diagonal operators on H_R admitting spectral synthesis the closure of whose eigenvalues $\{\lambda_n : n \ge 0\}$ have nonempty interior. This is in contrast to the case for diagonalizable operators on a separable complex Hilbert space (see Scroggs [18, Cor. 3.1, p. 104]).

It also follows from Theorem 4.4 that the coefficients $\{w_n\}$ in Wolff's example do not satisfy the condition that $\limsup |w_n|^{1/n} < 1$.

Finally, note that if $\{\lambda_n : n \ge 0\}$ is bounded, then $\inf(\{|\lambda_i - \lambda_j| : 0 \le i < j\}) = 0$. Nonetheless, since the set of cyclic diagonal operators are synthetic, they have as common cyclic vectors all functions with nonzero coefficients in their expansions by 4.1 condition 3. Hence, Theorem 4.4 is a companion to Theorem 3.5.

Theorem 4.5 *Let $D : H_1 \to H_1$ be a cyclic diagonal operator with associated sequence (λ_n) such that $(\frac{\lambda_n}{n})$ and $(\operatorname{Im}\lambda_n)$ are bounded sequences and $(\operatorname{Re}\lambda_n)$ is an increasing sequence. Then D is synthetic.*

Proof. Suppose that D is not synthetic. Then there is some non-zero sequence (γ_n) and $\varepsilon > 0$ such that $\limsup_{n\to\infty} |\gamma_n|^{\frac{1}{n}} < 1$, $\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}$ converges absolutely and uniformly on $B(0,\varepsilon)$, and $\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z} = 0$ on $B(0,\varepsilon)$. Note also that $\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}$ converges absolutely and uniformly on the set $G = \{z = a+bi : a \in (-\infty, 0), b \in (-1,1)\}$. To see this, let $K \subset G$ be compact, $R = \sup(\{\operatorname{Re} z : z \in K\}) < 0$, and $r = \inf(\{\operatorname{Re} z : z \in K\})$. Write $z = a + bi$ and $\lambda_n = a_n + b_n i$ for each $z \in K$ and all $n \ge 0$ and $A = \max(\{a_0 r, a_0 R\})$. By assumption, there is some M such that $|b_n| \le M$. Since (a_n) is an increasing sequence, for all $z \in K$ and

$n \geq 0$, we have that $\operatorname{Re} \lambda_n z = a_n a - b_n b \leq A + M$. Thus, for all $z \in K$, we have that

$$|\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}| \leq \sum_{n=0}^{\infty} |\gamma_n| e^{Re \lambda_n z} \leq e^{A+M} \sum_{n=0}^{\infty} |\gamma_n| < \infty.$$

Hence, $\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}$ converges absolutely and uniformly on G. Since $G \cap B(0, \varepsilon) \neq \emptyset$ and open and $\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z} = 0$ on $G \cap B(0, \varepsilon)$, then $\sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z} = 0$ on G. Let γ_{n_0} be such that $\gamma_{n_0} \neq 0$ and $\gamma_k = 0$ for $k < n_0$. Then $-\gamma_{n_0} = \sum_{n > n_0}^{\infty} \gamma_n e^{(\lambda_n - \lambda_{n_0}) z}$. Since (a_n) is an increasing sequence, we have that for $x < 0$

$$0 < |\gamma_{n_0}| \leq \sum_{n > n_0}^{\infty} |\gamma_n| |e^{(\lambda_n - \lambda_{n_0}) x}| \leq \sum_{n > n_0}^{\infty} |\gamma_n| e^{(a_n - a_{n_0}) x} \leq e^{(a_{n_0 + 1} - a_{n_0}) x} \sum_{n > n_0}^{\infty} |\gamma_n| \rightarrow 0$$

as $x \rightarrow -\infty$. Since this is a contradiction, D is synthetic. $\square$

Corollary 4.1 *The diagonal operator $D : H_1 \rightarrow H_1$ with associated sequence $(\lambda_n = n)$ is synthetic.*

It follows from Theorem 4.5 that the coefficients (w_n) in Natanson's example do not satisfy the condition that $\limsup_{n \to \infty} |w_n|^{\frac{1}{n}} < 1$.

Moreover, observe that if $\limsup_{n \to \infty} |\lambda_n|^{\frac{1}{n}} < 1$ then the set $\{\lambda_n : n \geq 0\}$ is bounded. Hence, when it comes to questions of synthesis, the only interesting diagonal operators are those which have associated sequences such that $\limsup_{n \to \infty} |\lambda_n|^{\frac{1}{n}} = 1$.

A final result for this chapter is the observation that the set of synthetic diagonal operators is dense in $\mathcal{D}$ in the SOT. This is the content of the next result.

Corollary 4.2 *The set of synthetic operators is dense in $\mathcal{D}$ in $C(H_1)$ in the SOT.*

Proof. Denote by $\mathcal{S}$ the set of synthetic, cyclic diagonal operators and let $D \in \mathcal{D}$ be given. Write $f(z) = \sum_{k=0}^{\infty} a_k z^k$ where $D = D_f$. Fix $n \geq 1$ and define $a_{n,0} = a_0$ and recursively define $a_{n,k} = a_k + \frac{m}{n(k+1)}$ where $m = \min(\{j : a_k + \frac{j}{n(k+1)} \notin \{a_{n,0}, \ldots, a_{n,k-1}\}$ and $0 \leq j \leq k\})$

for $0 \leq k \leq n$. Recursively choose $a_{n,k} \in [0,1] \setminus \{a_{n,j} : 0 \leq j \leq k\}$ for $k \geq n+1$. For $n \geq 1$ define $f_n \in H_1$ by $f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$.

Observe that each f_n has bounded distinct coefficients. Hence, D_{f_n} is a cyclic, synthetic operator. If $f_n \to f$ in H_1, then $D_{f_n} \to D_f = D$ in the SOT. To see that $f_n \to f$, observe that $\lim_{n\to\infty} a_{n,k} = a_k$ and that $\sup(\{|a_{n,k} - a_k|^{\frac{1}{k}} : k > 0\}) \leq \sup(\{n^{\frac{-1}{m}}, (|a_k| + 1)^{\frac{1}{k}} : 1 \leq m \leq n, k > n\})$. Since $\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} \leq 1$, $\limsup_{n\to\infty} \sup(\{|a_{n,k} - a_k|^{\frac{1}{k}} : k > 0\}) \leq 1$. Thus, $f_n \to f$ by Theorem 1.2. Therefore, $\overline{\mathcal{S}} = \mathcal{D}$. $\square$

CHAPTER 5

Polynomial Approximation And Synthesis

Note that the examples of synthetic diagonal operators in the last chapter both made use of condition 5 in Theorem 4.1. While this was useful in producing large classes of synthetic diagonal operators, it would also be nice to have some results concerning the synthesis of diagonal operators such that the sequence $\left(\frac{\lambda_n}{n}\right)$ is not bounded. For instance, is the operator with associated sequence $(\lambda_n = n^2)$ synthetic? In such cases, we can not use condition 5 of Theorem 4.1. In the absence of clever arguments involving some of the more abstract conditions in Theorem 4.1 or Theorem 4.3, we turn to more computational results. In particular, we consider condition 4 of Theorem 4.1 or condition 5 of Theorem 4.1. Since there is a large body of literature concerning polynomials, it would seem that condition 5 in Theorem 4.3 is a natural place to look.

Also, note that by Theorem 4.5 and condition 5 in Theorem 4.3, for each nonnegative integer j, there is some sequence of polynomials (p_n) such that $p_n(k) \to \delta_{j,k}$ and $\limsup_{n\to\infty} \sup\{|p_n(k)|^{\frac{1}{k}} : k > j\} \leq 1$. While we understand that such polynomials exist, the question becomes what do they look like? For instance, one may consider the sequence $p_n(z) = \prod_{k=1}^{n} \frac{z-k}{-k}$ or $q_n(z) = \prod_{k=1}^{n} \frac{(z-k)(z+k)}{-k^2}$ since $q_n \to \frac{\sin \pi z}{\pi z}$ in $H(\mathbb{C})$ (see [4] p. 175).

However, for $n \geq 2$ we have

$$|p_n(2n)| = \prod_{k=1}^{n} \frac{2n-k}{k} = \prod_{k=1}^{n} \frac{n+k-1}{k} = \prod_{k=1}^{n} 1 + \frac{n-1}{k} \geq \prod_{k=1}^{n} 1 + \frac{n-1}{n} = \left(\frac{3}{2}\right)^n$$

and

$$q_n(n+1) = \left(\prod_{k=1}^{n} \frac{n+1-k}{k}\right)\left(\prod_{k=1}^{n} \frac{n+1+k}{k}\right) = \prod_{k=1}^{n} 1 + \frac{n+1}{k} \geq \prod_{k=1}^{n} 1 + \frac{n+1}{n} \geq 2^n.$$

Thus, for any $j \geq 0$, $\limsup_{n\to\infty} \sup(\{|p_n(k)|^{\frac{1}{k}} : k > j\}) \geq \left(\frac{3}{2}\right)^{\frac{n}{2n}} > 1$ and $\limsup_{n\to\infty} \sup(\{|q_n(k)|^{\frac{1}{k}} : k > j\}) \geq 2$. Hence, the obvious choices for the appropriate polynomials do not work. So what polynomials do work?

In the first section of this chapter, we will prove some results which help us to determine the zeros of the polynomials that possess the prescribed behavior. In the second section, we will provide an example from which to, it is hoped, extrapolate techniques to answer questions about synthesis for diagonal operators for which the sequence $\left(\frac{\lambda_n}{n}\right)$ is not bounded.

5.1 Preliminaries

Let us make a few simple observations. To this end, let $D : H_1 \to H_1$ be a cyclic, diagonal operator with associated sequence (λ_n). First, by Proposition 4.2, we may assume, without loss of generality, that $\lambda_0 = 0$. Next, suppose that there is some sequence $(p_n) \subset \mathbb{C}[z]$ such that $p_n(\lambda_k) \to \delta_{0,k}$ and $\limsup_{n\to\infty} \sup_{k>0} |p_n(\lambda_k)|^{\frac{1}{k}} \leq 1$. Since $p_n(0) \to 1$, we may suppose without loss of generality, that $p_n(0) \neq 0$ for all $n \geq 1$. Hence, we may define $q_n(z) = \frac{1}{p_n(0)} p_n(z)$ and observe that $q_n(\lambda_k) \to \delta_{0,k}$, $\limsup_{n\to\infty} \sup_{k>0} |q_n(\lambda_k)|^{\frac{1}{k}} \leq 1$, and $q_n(0) = 1$ for all n. Thus, in our attempts to construct sequences of polynomials, we need only focus our attention on polynomials of the form $\prod_{k=1}^{n} \frac{z-z_k}{-z_k}$. Third, if $x > 0$ and $\operatorname{Re} z < 0$, then $\frac{|x-z|^2}{|z|^2} = \frac{(x-a)^2+b^2}{a^2+b^2} \geq \frac{a^2+b^2}{a^2+b^2} = 1$ where $z = a + bi$. Hence, if we have positive eigenvalues and are considering polynomials of the aforementioned form, we will assume that $\operatorname{Re} z_k > 0$.

Lemma 5.1 *For* $w, z \in \mathbb{C}$, $\left|\frac{w-z}{-z}\right| = \left(1 + \frac{-2\operatorname{Re}(w\overline{z})+|w|^2}{|z|^2}\right)^{\frac{1}{2}}$.

Proof. Observe that

$$\left|\frac{w-z}{-z}\right|^2 = \frac{(z-w)(\overline{z}-\overline{w})}{|z|^2} = \frac{|z|^2 - 2\operatorname{Re}(w\overline{z}) + |w|^2}{|z|^2} = 1 + \frac{-2\operatorname{Re}(w\overline{z}) + |w|^2}{|z|^2}.$$

□

Proposition 5.1 *Let* $(z_n) \subset \mathbb{C}$ *such that* $0 < \operatorname{Re}(z_n) \to \infty$ *and* $x_0 > 0$ *such that* $x_0 \notin \{z_n : n \geq 1\}$ *be given. Define* $p_n(x) = \prod_{k=1}^{n} \frac{x-z_k}{-z_k}$. *Then* $\lim_{n\to\infty} p_n(x_0) = 0$ *if and only if* $\sum_{k=1}^{\infty} \frac{\operatorname{Re}(z_n)}{|z_k|^2} = \infty$.

Proof. If $x_0 \notin \{z_n : n \geq 1\}$, then $\lim_{n\to\infty} |p_n(x_0)| = 0$ if and only if $\prod_{n=1}^{\infty} \frac{|x_0 - z_n|}{|-z_n|} = 0$ if and only if $\sum_{n=1}^{\infty} \ln \frac{|x_0-z_n|}{|-z_n|} = -\infty$. Further, by Lemma 5.1, $p_n(x_0) \to 0$ if and only if

$$-\infty = \sum_{n=1}^{\infty} \frac{1}{2} \ln\left(1 + \frac{-2\operatorname{Re}(x_0\overline{z}_n) + x_0^2}{|z_n|^2}\right) = \sum_{n=1}^{\infty} \frac{1}{2} \ln\left(1 + x_0\frac{-2\operatorname{Re}(z_n) + x_0}{|z_n|^2}\right).$$

Since $\operatorname{Re}(z_n) \to \infty$, there is some N such that $-2\operatorname{Re}(z_n) + x_0 < 0$ and $x_0\frac{2\operatorname{Re}(z_n)-x_0}{|z_n|^2} = \left|x_0\frac{-2\operatorname{Re}(z_n)+x_0}{|z_n|^2}\right| < \frac{1}{2}$ for $n \geq N$. Thus, by Equation 5.3 on p. 165 in [4] for $n \geq N$,

$$\begin{aligned}\frac{1}{2}x_0\frac{2\operatorname{Re}(z_n) - x_0}{|z_n|^2} &\leq -\ln\left(1 + x_0\frac{-2\operatorname{Re}(z_n) + x_0}{|z_n|^2}\right) \\ &= \left|\ln\left(1 + x_0\frac{-2\operatorname{Re}(z_n) + x_0}{|z_n|^2}\right)\right| \leq \frac{3}{2}x_0\frac{2\operatorname{Re}(z_n) - x_0}{|z_n|^2}.\end{aligned}$$

Hence,

$$-\frac{3}{4}x_0\frac{2\operatorname{Re}(z_n) - x_0}{|z_n|^2} \leq \ln\frac{|x_0 - z_n|}{|-z_n|} \leq -\frac{1}{4}x_0\frac{2\operatorname{Re}(z_n) - x_0}{|z_n|^2}$$

for $n \geq N$. This implies that $\sum_{n=1}^{\infty} \ln \frac{|x_0-z_n|}{|-z_n|} = -\infty$ if and only if $\sum_{n=N}^{\infty} x_0\frac{2\operatorname{Re}(z_n)-x_0}{|z_n|^2} = x_0\sum_{n=N}^{\infty} \frac{2\operatorname{Re}(z_n)-x_0}{|z_n|^2} = \infty$. Clearly, this happens if and only if $\sum_{n=1}^{\infty} \frac{\operatorname{Re}(z_n)}{|z_n|^2} = \infty$. □

Proposition 5.2 *Suppose that* $(z_n) \subset \mathbb{R}$ *be such that* $z_n \uparrow \infty$ *and* $z_1 \geq \frac{e}{2}$. *Define* $p_n(x) = \prod_{j=1}^{n} \frac{x - z_j}{-z_j}$. *Then* $\sup_{x > x_0} |p_n(x)|^{\frac{1}{x}} \leq \max(\{(2z_k)^{\frac{k}{2z_k}}, (2z_{k+1})^{\frac{k+1}{2z_{k+1}}}, \ldots, (2z_n)^{\frac{n}{2z_n}}\})$ *where* $x_0 \in (2z_k, 2z_{k+1}]$.

Proof. Observe that $\frac{|x-z_j|}{z_j} \leq 1$ when $0 \leq x \leq 2z_j$. Next, note that if $x > 2z_j$, then since $z_j \geq \frac{e}{2} > 1$, we have $\frac{|x-z_j|}{z_j} = \frac{x}{z_j} - 1 \leq x$. Hence, if $2z_k < x \leq 2z_{k+1}$, then $|p_n(x)| = \prod_{j=1}^{n} \frac{|x-z_j|}{z_j} \leq \prod_{j=1}^{k} \frac{|x-z_j|}{z_j} \leq x^k$. Similarly, when $x > 2z_n$, $|p_n(x)| = \prod_{j=1}^{n} \frac{|x-z_j|}{z_j} \leq x^n$. Thus, $|p_n(x)|^{\frac{1}{x}} \leq x^{\frac{k}{x}}$ when $2z_k < x \leq 2z_{k+1}$ and $|p_n(x)|^{\frac{1}{x}} \leq x^{\frac{n}{x}}$ when $x > 2z_n$. Observe that the function $g(x) = x^{\frac{1}{x}}$ achieves its maximum at $x = e$, decreases on $[e, \infty)$, and $2z_k \geq e$ for all $k \geq 1$. Hence,

$$\sup_{2z_k < x \leq 2z_{k+1}} |p_n(x)|^{\frac{1}{x}} \leq \sup_{2z_k < x \leq 2z_{k+1}} x^{\frac{k}{x}} = (2z_k)^{\frac{k}{2z_k}}$$

and

$$\sup_{x > 2z_n} |p_n(x)|^{\frac{1}{x}} \leq \sup_{x > 2z_n} x^{\frac{n}{x}} \leq (2z_n)^{\frac{n}{2z_n}}.$$

Therefore, $\sup_{x > x_0} |p_n(x)|^{\frac{1}{x}} \leq \max(\{(2z_k)^{\frac{k}{2z_k}}, \ldots, (2z_n)^{\frac{n}{2z_n}}\})$ and the theorem is proven. $\square$

5.2 Proof Of Synthesis By Polynomial Approximation

Using condition 5 in Theorem 4.3 and the preceding propositions, we now present a constructive argument that the operator $D : H_1 \to H_1$ with associated sequence $(\lambda_n = n)$ is synthetic.

Theorem 5.1 *Let* $D : H_1 \to H_1$ *be the diagonal operator with associated sequence* $(\lambda_n = n)$. *The operator* D *is synthetic.*

Proof. Write $z_k = (k+3)\ln(k+3)\ln(\ln(k+3))$. Note that since

$$\sum_{k=3}^{\infty} \frac{1}{k \ln(k) \ln(\ln k)} \geq \int_{3}^{\infty} \frac{1}{x \ln(x) \ln\ln x} dx = \int_{\ln 3}^{\infty} \frac{1}{x \ln x} dx = \int_{\ln\ln 3}^{\infty} \frac{1}{x} = \infty, \tag{5.1}$$

$\sum_{n=1}^{\infty} \frac{1}{(k+3)\ln(k+3)\ln(\ln(k+3))} = \infty$. Define $p_n(x) = \prod_{k=1}^{n} \frac{x-z_k}{-z_k}$, and note that $z_1 > \frac{e}{2}$. Thus, from Proposition 5.2, $\lim_{n\to\infty} p_n(x) = \delta_{0,x}$ for $x \geq 0$. Define

$$\begin{aligned} g(x) &= (2(x+3)\ln(x+3)\ln(\ln(x+3)))^{\frac{1}{2\ln(x+3)\ln(\ln(x+3))}} \\ &= (2\ln(x+3))\ln(\ln(x+3)))^{\frac{1}{2\ln(x+3)\ln(\ln(x+3))}} \cdot (x+3)^{\frac{1}{2\ln(x+3)\ln(\ln(x+3))}} \end{aligned}$$

for $x > 0$. Since $\frac{d}{dx}\frac{1}{2\ln(x+3)\ln(\ln(x+3))}\ln(x+3) < 0$ and $(2\ln(x+3)\ln(\ln(x+3)))^{\frac{1}{2\ln(x+3)\ln(\ln(x+3))}}$ decreases when $2\ln(x+3)\ln(\ln(x+3)) > e$, we have that g is decreasing on (M, ∞) where M is sufficiently large. Moreover, since $\ln g(x) = \frac{2+\ln(x+3)+\ln(\ln(x+3))+\ln(\ln(\ln(x+3)))}{2\ln(x+3)\ln(\ln(x+3))} \to 0$ as $x \to \infty$, $g(x) \to 1$ as $x \to \infty$. Also,

$$\begin{aligned} (2z_k)^{\frac{k}{2z_k}} &= (2(k+3)\ln(k+3)\ln(\ln(k+3)))^{\frac{k}{2(k+3)\ln(k+3)\ln(\ln(k+3))}} \\ &\leq (2(k+3)\ln(k+3)\ln(\ln(k+3)))^{\frac{1}{2\ln(k+3)\ln(\ln(k+3))}} = g(k). \end{aligned}$$

Choose $K \in \mathbb{N}$ such that $g(K) < 1+\varepsilon$ and $K \geq M$. Since $p_n(k) \to 0$ as $n \to \infty$ for $k \geq 1$, there exists an $N \geq K$ such that $|p_n(k)| \leq 1$ for $1 \leq k \leq 2(K+4)\ln(K+4)\ln(\ln(K+4)) = 2z_{K+1}$ and $n \geq N$. Then by Proposition 5.2, for $n \geq N$, we have that

$$\begin{aligned} \sup_{k>1} |p_n(k)|^{\frac{1}{k}} &= \sup_{k>z_{K+1}} |p_n(k)|^{\frac{1}{k}} \leq \sup_{x>z_{K+1}} |p_n(x)|^{\frac{1}{x}} \leq \max(\{(2z_K)^{\frac{K}{2z_K}}, \ldots, (2z_n)^{\frac{n}{2z_n}}\}) \\ &\leq \max(\{g(K), \ldots, g(n)\}) = g(K) < 1+\varepsilon. \end{aligned}$$

Thus, $\limsup_{n\to\infty} \sup_{k>1} |p_n(k)|^{\frac{1}{k}} \leq 1$.

Let $j \in \mathbb{N}$ be given and define $q_n(x) = \prod_{m=0}^{j-1} \frac{x-m}{j-m} p_n(x-j)$. By construction, $\lim_{n\to\infty} q_n(k) = \delta_{j,k}$. Also, we have for $k > j$ that

$$|q_n(k)|^{\frac{1}{k}} \leq k^{\frac{j}{k}}(|p_n(k-j)|^{\frac{1}{k-j}})^{\frac{k-j}{k}} \leq k^{\frac{j}{k}} \max(\{|p_n(k-j)|^{\frac{1}{k-j}}, 1\}).$$

Let $\varepsilon > 0$ be given and choose $K \in \mathbb{N}$ such that $K \geq \max(\{e, j\})$ and $K^{\frac{j}{k}} < \sqrt{1+\varepsilon}$. Choose N such that $\sup(\{|p_n(k-j)|^{\frac{1}{k-j}} : k > j\}) < \sqrt{1+\varepsilon}$ and $|q_n(k)| \leq 1$ for $j+1 \leq k \leq K$. Then for $n \geq N$, we have that $\sup(\{|q_n(k)|^{\frac{1}{k}} : k > j\}) < 1+\varepsilon$. Therefore, $\limsup_{n\to\infty} \sup(\{|q_n(k)|^{\frac{1}{k}} : k > 0\}) \leq 1$ and D is synthetic by condition 5 of Theorem 4.3. $\square$

5.3 Concluding Remarks

In this document, it was shown that $H(G)$ embeds in $C(H(G))$ (Theorem 1.5 and Proposition 2.4). In Chapter 3, we classified which diagonal operators are cyclic (Theorem 3.1), showed that cyclicity is heavily related to the decay rate of coefficients of a vector (Proposition 3.2 and Theorem 3.4), and demonstrated the existence of common cyclic vectors for a given family of diagonal operators (Theorem 3.5). Theorem 4.1 and Theorem 4.3 give a number of equivalent conditions for a diagonal operator to be synthetic while Theorem 4.4 and Theorem 4.5 demonstrate that diagonal operators with certain restrictions on their growth rates are synthetic. Finally, in Theorem 5.1 we constructed polynomials whose existence was guaranteed by Theorem 4.3 and Theorem 4.5. With these results in mind, it seems that there are least four places to continue researching.

First, observe that if $L \in H_1^*$ and $\ell_n = L(z^n)$, then $\limsup_{n\to\infty} |\ell_n|^{\frac{1}{n}} <$ and $f \in H_1$ where $f(z) = \sum_{n=0}^{\infty} \ell_n z^n$. Hence, in an algebraic way, $H_1^* \subseteq H_1$. That is, H_1 has a similar relationship to its dual that a Hilbert space has. It may be possible to exploit this to learn more about which cyclic diagonal operators are synthetic. To see this, recall from Theorem 4.3 that if D is a cyclic diagonal operator and $\mathcal{A}$ is the algebra generated by D, then $\overline{\mathcal{A}} = \mathcal{D}$ in the SOT if and only if D is synthetic. Also observe that since $\mathcal{D}$ is closed and $DD' = D'D$ for all $D' \in \mathcal{D}$, that $\mathcal{D}$ is the double commutant of $\mathcal{A}$. Thus, it may be possible to prove some type of analogue of the double communtant theorem for $C(H_1)$. The real work in such a task would be to define a suitable topology on H_1^*. In particular, one should define the topology on H_1^* in such a way that adjoint operators are continuous. This would serve to extend the work in section 1.5 of this document.

Second, notice that issues of decay rates of coefficients of functions as well as eigenvalues of operators appear frequently. These should be explored further. In particular, in light of condition 4 in Theorem 4.3 and Corollary 3.1, a conjecture that should investigated is the following. Suppose D and D' are cyclic diagonal operators with nonnegative, real, unbounded, and increasing associated sequences (λ_n) and (λ'_n) respectively. If $\lambda'_n \leq \lambda_n$ for $n \geq 0$ and D is synthetic, is D' synthetic? Similarly, if $\lambda'_n \geq \lambda_n$ for $n \geq 0$ and D is not synthetic, is D' not synthetic? Another conjecture along these lines is that given a non-synthetic operator, it is possible to construct another non-synthetic operator whose eigenvalues grow faster and which has fewer cyclic vectors than the first operator.

Third, determining whether or not there are common cyclic vectors for all diagonal operators should continue to be investigated. Note that from Theorem 3.5, all cyclic diagonal operators whose eigenvalues are separated have a residual set of common cyclic vectors. Observe that given an infinite set $\{\lambda_n : n \geq 0\}$ such that $\inf(\{|\lambda_m - \lambda_n : m \neq n\}) > 0$, $\{\lambda_n : n \geq 0\}$ must be unbounded. Hence, it would be natural to wonder whether or not diagonal operators with bounded eigenvalues have common cyclic vectors. However, that is guaranteed by Theorem 4.5 and condition 3 in Theorem 4.1. Thus, the family of cyclic diagonal operators with bounded eigenvalues or separated eigenvalues has a residual set of common cyclic vectors. By condition 3 in Theorem 4.1, if there is no common cyclic vector to all cyclic diagonal operators, then there exists an non-synthetic cyclic diagonal operator. It would be worth considering if the converse is true. This may be connected to the conjectures in the preceding paragraph.

Finally, observe that the estimates in the argument of Proposition 5.2 were not nearly as sharp as they could be. It may be possible to generalize the construction in chapter 5 to perhaps show that some diagonal operator D with associated sequence (λ_n) such that the set $\left\{\frac{\lambda}{n} : n \geq 1\right\}$ is not bounded is synthetic. It may also be worthwhile constructing polynomials which show that a diagonal operator with bounded eigenvalues is synthetic.

BIBLIOGRAPHY

[1] L. Bernal-González, M. C. Calderón-Moreno*, J. A. Prado-Bassas, "Cyclicity of coefficient multipliers: linear structure", *Acta Mathematica Hungarica*, Springer Netherlands, vol. 114 (2006), no. 4, pp. 287-300.

[2] L. Brown, A. Shields, K. Zeller, "On absolutley convergent exponential sums," *Trans. Amer. Math. Soc.*, vol. 96 (1960), 162 - 183.

[3] J. Conway, A Course In Functional Analysis 2nd Edition, Springer, 1990.

[4] J. Conway, Functions of One Complex Variable 2nd Edition, Springer-Verlag, 1978.

[5] I. Deters, S. Seubert, "Cyclic vectors of diagonal operators on the space of functions analytic on a disk", *J. Math. Anal. Appl.* vol. 334 (2007) 1209-1219.

[6] K. Grosse-Erdmann, "Universal families and hypercyclic operators", *Bulletin Of The American Mathmatical Society* Vol. 36 (1999), no. 3, 345 - 381.

[7] D. Herrero, "Possible structures for the set of cyclic vectors," *Indiana U. Math. J.*, vol. 6 (1979). 913-926.

[8] V. G. Iyer, "On the space of integral functions-I", *J. Indian Math. Soc.* vol. 12 (1948), 13-30.

[9] J. P. Lesko, S. M. Seubert, "Cyclicity results for jordan and compressed toeplitz operators". *Integr. equ. oper. theory* vol. 31 (1998), no. 3, 338 - 352

[10] J. Marin Jr., S. M. Seubert, "Cyclic vectors of diagonal operators on the space of entire functions". *J. Math. Anal. Appl.* vol. 320 (2006) 599 - 610.

[11] R. Megginson, An Introduction To Banach Space Theory, Springer, 1998.

[12] N. K. Nikol' skiĭ, Operators, Functions, and Systems: An Easy Reading, Vol. I Mathematical surveys and monographs, Vol. 92, American Mathematical Society, Providence, RI, 2002.

[13] N. K. Nikol' skiĭ, "The present state of of the spectral analysis-synthesis problem I" *in* "Fifteen Papers on Functional Analysis" (Amer. Math. Soc. Transl., (1984), 97 - 129, v. 124, Providence, RI).

[14] W. T. Ross and H. S. Shapiro, Generalized Analytic Continuation, Vol. 25, University Lecture Series, American Mathematical Society, Providence, RI, 2002.

[15] W. Rudin, Functional Analysis 2nd Edition, McGraw-Hill, 1991.

[16] D. Sarason, "Invariant subspaces and unstarred operators algebras," *Pacific J. Math.*, vol. 17 (1966), 511-517.

[17] D. Sarason, "Weak-star density of polynomials," *J. Reine Angew. Math.*, vol. 252 (1972), 1-15.

[18] J. E. Scroggs, "Invariant subspaces of normal operators, " *Duke Math. J.*, vol. 26 (1959), 95-112.

[19] A. Shields, "Weighted shift operators and analytic function theory" *in* Mathematical Surveys, (Amer. Math. Soc., (1974), 51-128, vol. 13, Providence, RI).

[20] R.V. Sibilev, "Uniqeness theorem for Wolff-Denjoy series," *Algebra i Analiz*, vol. 7 (1995), 170–199. English Translation in *St. Petersburg Math. J.*, vol. 7 (1996), 145-168.

[21] J. Wermer, “On invariant subspaces of normal operators,” *Proc. Amer. Math. Soc.*, vol. 3 (1952), 270-277.

[22] J. Wolff, “Sur les series $\sum A_k/(z - z_k)$”, *Comptes Rendus*, vol 173 (1921), 1057-1058, 1327-1328.