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Research Statement

The form and invariant subspaces of linear operators on $\mathbb{C}^n$ can be deduced using the Jordan Decomposition Theorem. One current line of research is to study Jordan like operators on infinite dimensional spaces and try to determine their invariant subspaces. For instance, finding the invariant subspaces of the backwards shift on ℓ^2 had its celebrated solution in Beurling's theorem. The shift operator on ℓ^2 can be thought of as an infinite by infinite matrix with a single, large cell. On the other end of the spectrum of Jordan like operators are the diagonal operators. These may be visualized as infinite by infinite matrices with one by one cells. I research diagonal operators on the space of functions analytic on the unit disc. In particular, I am concerned with the closed invariant subspaces of such operators. Recently, this has brought me to consider the existence of sequences of polynomials which seperate and have a minimal type of growth condition on the eigenvalues of such operators.

More precisely, let $H_1 \equiv H(B(0,1))$ and $C(H_1)$ denote the space of functions analytic on the unit disc in $\mathbb{C}$ and the set of continuous linear operators on H_1, respectively. The topology on H_1 is that of uniform convergence on compact subsets of $B(0,1)$. The topology on $C(H_1)$ is the topology of pointwise convergence. This will be referred to as the strong operator topology or SOT. A continuous map $D : H_1 \to H_1$ given by $D\sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \lambda_n f_n z^n$ is called a diagonal operator. Given a diagonal operator D, the set $\{\lambda_n : n \geq 0\}$ is its set of eigenvalues and $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq 1$. Observe that this means that it is possible to have unbounded eigenvalues. Let $\mathcal{D}$ denote the set of diagonal operators on H_1. The set $\mathcal{D} \subseteq C(H_1)$ is closed and is an embedding of H_1 into $C(H_1)$.

A continuous operator T on a complete metrizable space X is *cyclic* if there is some $x \in X$ such that $\overline{\text{span}(\{T^n x : n \geq 0\})} = \overline{\{p(T)x : p \in \mathbb{C}[z]\}} = X$. In this case, the vector x is said to by cyclic for T. Cyclic vectors are related to the invariant subspaces of T, since the set of cyclic vectors is equal to the complement of the union of all non-trivial closed invariant subspaces. An operator T is *synthetic* if every closed invariant subspace of T is the closure of the span of some set of T's eigenvectors. Since the closure of the span of some set of T's eigenvectors is a closed invariant subspace, synthesis is a type of minimality condition on T's closed invariant subspaces.

I have shown that a diagonal operator is cyclic if and only if it has distinct eigenvalues. In this case, it has a dense set of cyclic vectors. Also, if a vector $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is to be cyclic, then $f_n \neq 0$ for all $n \geq 0$. I have also studied when cyclic diagonal operators are synthetic. Some of the more interesting equivalences for a cyclic diagonal operator $D : H_1 \to H_1$ given by $D\sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \lambda_n f_n z^n$ to be synthetic are as follows:

1. The operator D is synthetic.
2. Every function $f \in H_1$ such that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $f_n \neq 0$ for all $n \geq 0$ is cyclic for D.

3. The function $u(z) = \frac{1}{1-z}$ is cyclic for D.

4. For each $j \geq 0$ there is some sequence $(p_n) \subseteq \mathbb{C}[z]$ such that $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\limsup_{n\to\infty} \sup_{k>j} |p_n(\lambda_k)|^{\frac{1}{k}} \leq 1$

5. If $\mathcal{A}$ is the algebra generated by D and the identity, then $\overline{\mathcal{A}} = \mathcal{D}$.

6. There does not exist a non-trivial sequence $(w_n) \subseteq \mathbb{C}$ for which $\limsup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$.

7. If, in addition, $\left\{\frac{\lambda_n}{n} : n \geq 1\right\}$ is bounded, then $g \in H_\varepsilon$ where $g(z) = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ whenever $(w_n) \subseteq \mathbb{C}$ for which $\limsup |w_n|^{1/n} < 1$ and $\epsilon = [\ln(1/\limsup |w_n|^{1/n})]/[\sup(\{|\lambda_n|/n : n \geq 1\})]$. There does not exist a non-trivial sequence $(w_n) \subseteq \mathbb{C}$ for which $\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < 1$ and $0 = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ for all $z \in B(0, \epsilon)$.

Condition 3 is interesting since it indicates that knowing that a single vector is cyclic, allows one to know all of the closed invariant subspaces of the operator. This is in contrast to the general case where one needs to know all of the cyclic vectors in order to deduce the closed invariant subspaces.

Observe that if the set $\{\lambda_n : n \geq 0\}$ is unbounded and $p \in \mathbb{C}[z]$ and is non - constant, then $\sup_{k>j} |p(\lambda_k)|^{\frac{1}{k}} > 1$. Thus, the polynomials in condition 4 satisfy a minimal type of growth condition. Condition 4 is also nice because it gives a purely computational way to explore whether or not a diagonal operator is synthetic. It also has the advantage that it can potentially make use of the literature concerning polynomials.

Using condition 7, I have shown that diagonal operators with bounded eigenvalues are synthetic. I have also shown that diagonal operators with unbounded eigenvalues which satisfy various growth conditions are synthetic.

Note that the preceding paragraph implies that all diagonal operators with bounded eigenvalues have a dense set common cyclic vectors. I have also shown that the family of diagonal operators with seperated eigenvalues have a dense set of common cyclic vectors.

I see four places for further research. First, my research combined with the research of Kate Overmoyer and Melanie Henthorn has demonstrated that the synthesis of a diagonal operator depends on both the rate of growth of the eigenvalues of the operator as well as how they are spread around the plane. However, the precise nature of this dependency is not clear. Investigating the nature of the interplay between these two factors as well as determining an upper bound on the growth rate of the eigenvalues to ensure synthesis would surely make for good research.

Second, there is the matter of constructing the polynomials who existence is guarenteed in condition 4. Most recently I have constructed polynomials which show that a family of diagonal operators with associated sequences (λ_n) such that the set $\left\{\frac{|\lambda_n|}{n} : n \geq 1\right\}$ is not bounded are synthetic. It may also be worthwhile constructing the polynomials which show that a diagonal operator with bounded eigenvalues is synthetic.

Third, one may show that $H_1^* \subseteq H_1$ with an appropriate interpretation. That is, H_1 has a similar relationship to its dual that a Hilbert space has. It may be possible to exploit this to learn more about which cyclic diagonal operators are synthetic. To see this, recall from condition 5 that if D is a cyclic diagonal operator and $\mathcal{A}$ is the algebra generated by D, then $\overline{\mathcal{A}} = \mathcal{D}$ in the SOT if and only if D is synthetic. Also, $\mathcal{D}$ is the double commutant of $\mathcal{A}$. Thus, it may be possible to prove some type of analogue of the double communtant theorem for $C(H_1)$. The real work in such a task would be to define a suitable topology on H_1^*. In particular, one should define the topology on H_1^* in such a way that adjoint operators are continuous.

Finally, determining whether or not there are common cyclic vectors for all diagonal operators should be investigated. While the work of Henthorn and Overmoyer shows there exists a non-synthetic, cyclic, diagonal operator, there is still the possbility that such vectors exist. Producing such a vector or demonstrating its non-existence has not yet been accomplished.